CHAPTER 19

Random Processes

19.1 BASIC DEFINITIONS

In the earlier sections a random variable X was defined as a function that maps every outcome ξ_i of points in the sample space S to a number $X(\xi_i)$ on the real line R. A *random process* $X(t)$ is a mapping that assigns a time function $X(t,\xi_i)$ to every outcome ξ_i of points in the sample space S. Alternate names for random processes are *stochastic processes* and *time series*. More formally, a random process is a time function assigned for *every* outcome $\xi \in S$ according to some rule $X(t,\xi)$, $t \in T$, $\xi \in S$, where T is an index set of time. As in the case of a random variable, we suppress ξ and define a random process by $X(t)$. If the index set T is countably infinite, the random process is called a discrete-time process and is denoted by X_n.

Referring to Fig. 19.1.1, a random process has the following interpretations:

1. $X(\xi,t_1)$ is random variable for a fixed time t_1.
2. $X(\xi_i,t)$ is a *sample realization* for any point ξ_i in the sample space S.
3. $X(\xi_i,t_2)$ is a number.
4. $X(\xi,t)$ is a collection or *ensemble* of realizations and is called a *random process*.

An important point to emphasize is that a random process is a finite or an infinite ensemble of time functions and is not a single time function.

Example 19.1.1 A fair coin is tossed. If heads come up, a sine wave $x_1(t) = \sin(5\pi t)$ is sent. If tails come up, then a ramp $x_2(t) = t$ is sent. The resulting random process $X(t)$ is an ensemble of two realizations, a sine wave and a ramp, and is shown in Fig. 19.1.2. The sample space S is discrete.

Example 19.1.2 In this example a sine wave is in the form $X(t) = A\sin(\omega t + \Phi)$, where Φ is a random variable uniformly distributed in the interval $(0, 2\pi)$. Here the sample space is continuous, and the sequence of sine functions is shown in Fig. 19.1.3.

Probability and Random Processes, by Venkatarama Krishnan

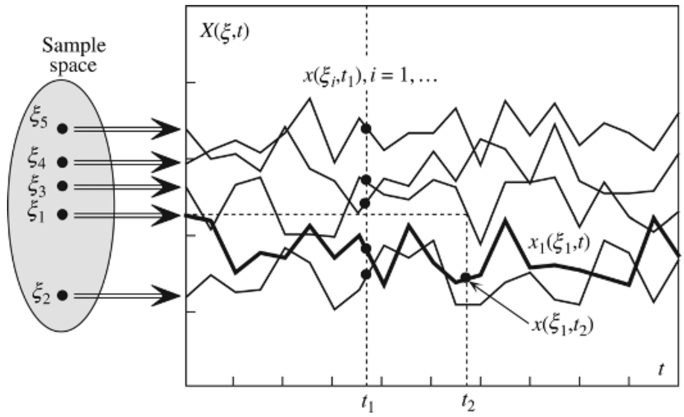


FIGURE 19.1.1

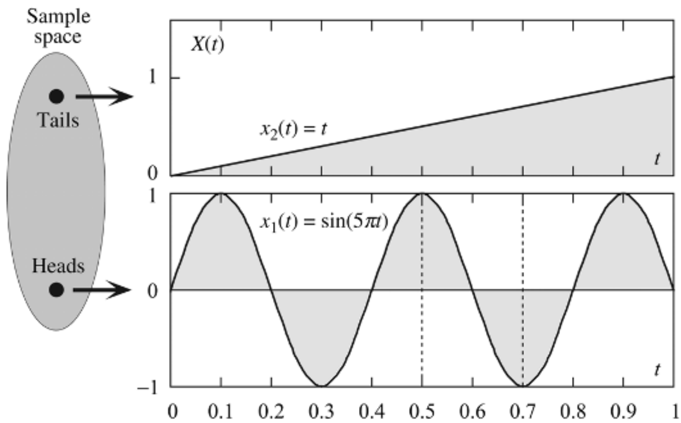


FIGURE 19.1.2

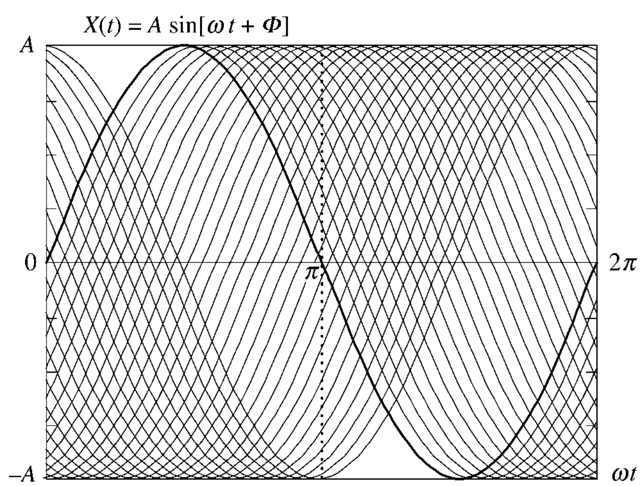


FIGURE 19.1.3

Distribution and Density Functions. Since a random process is a random variable for any fixed time t, we can define a probability distribution and density functions as

$$F_X(x;\, t) = P(\xi,\, t\colon X(\xi;\, t) \le x) \text{ for a fixed } t \tag{19.1.1}$$

and

$$\begin{aligned} f_X(x;\, t) = \frac{\partial}{\partial x} F_X(x;\, t) &= \lim_{\Delta x \to 0} \frac{F_X(x + \Delta x;\, t) - F_X(x;\, t)}{\Delta x} \\ &= \lim_{\Delta x \to 0} P(x < X(t) \le x + \Delta x) \end{aligned} \tag{19.1.2}$$

These are also called *first-order* distribution and density functions, and in general, they are functions of time.

Means and Variances. Analogous to random variables, we can define the mean of a random process as

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;\, t)dx \tag{19.1.3}$$

and the variance as

$$\begin{aligned} \sigma_X^2(t) &= E[X(t) - \mu_X(t)]^2 = E[X^2(t)] - \mu_X^2(t) \\ &= \int_{-\infty}^{\infty} [x - \mu_X(t)]^2 f_X(x;\, t)dx \end{aligned} \tag{19.1.4}$$

where

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x;\, t)dx \tag{19.1.5}$$

Since the density is a function of time, the means and variances of random processes are also functions of time.

Example 19.1.3 We shall now find the distribution and density functions along with the mean and variance for the random process of Example 19.1.1 for times $t = 0,\ \frac{1}{2},\ \frac{7}{10}$:

$$t = 0, \quad x_1(0) = 0, \quad x_2(0) = 0$$

At $t = 0$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4a along with the corresponding distribution and density functions.

The mean value is given by $\mu_X(0) = 0;\ \frac{1}{2} + 0 \cdot \frac{1}{2} = 0$. The variance is given by $\sigma_X^2(0) = (0-0)^2\frac{1}{2} + (0-0)^2\frac{1}{2} = 0$:

$$t = \frac{1}{2}, \quad x_1\left(\frac{1}{2}\right) = 1, \quad x_2\left(\frac{1}{2}\right) = \frac{1}{2}$$

At $t = \frac{1}{2}$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4b along with the corresponding distribution and density functions.

The mean value is given by $\mu_X\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4} = 0.75$.

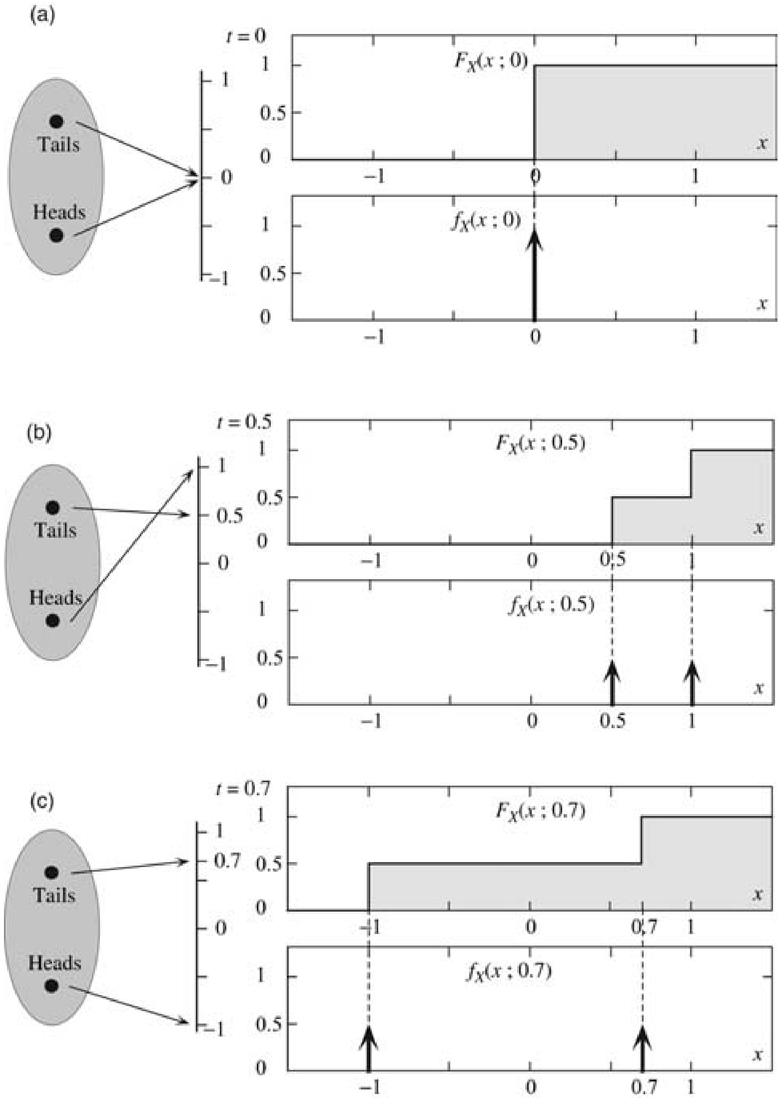


FIGURE 19.1.4

The variance is given by $\sigma_X^2\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \frac{3}{4}\right)^2 \frac{1}{2} + \left(1 - \frac{3}{4}\right)^2 \frac{1}{2} = \frac{1}{16} = 0.0625$:

$$t = \frac{7}{10}, \quad x_1\left(\frac{7}{10}\right) = 1, \quad x_2\left(\frac{7}{10}\right) = \frac{7}{10}$$

At $t = \frac{7}{10}$ the mapping diagram from the sample space to the real line is shown in Fig. 19.1.4c along with the corresponding distribution and density functions.

The mean value is given by $\mu_X\left(\frac{7}{10}\right) = \frac{7}{10} \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = -\frac{3}{20} = -0.15$
The variance is given by $\sigma_X^2\left(\frac{7}{10}\right) = \left(\frac{7}{10} + \frac{3}{20}\right)^2 \frac{1}{2} + \left(-1 + \frac{3}{20}\right)^2 \frac{1}{2} = \frac{289}{400} = 0.7225.$

Example 19.1.4 A random process, given by $X(t) = A \sin(\omega t)$, is shown in Fig. 19.1.5, where A is a random variable uniformly distributed in the interval (0,1].

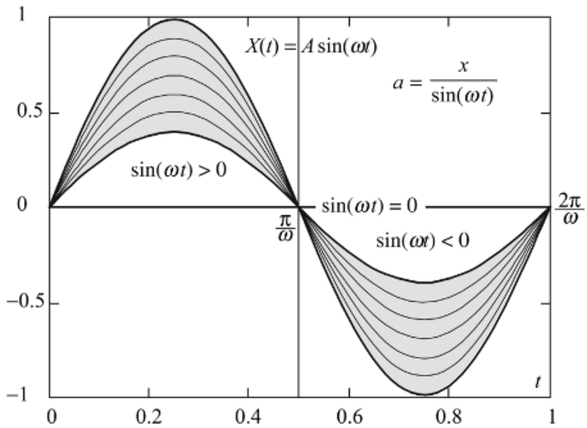


FIGURE 19.1.5

The density and distribution functions of A are

$$f_A(a) = \begin{bmatrix} 1, & 0 < a \leq 1 \\ 0, & \text{otherwise} \end{bmatrix} \quad F_A(a) = \begin{bmatrix} 0, & a \leq 0 \\ a, & 0 < a \leq 1 \\ 1, & a > 1 \end{bmatrix}$$

We have to find the distribution function $F_X(x;\, t)$. For any given t, $x = a\sin(\omega t)$ is an equation to a straight line with slope $\sin(\omega t)$, and hence we can use the results of Examples 12.2.1 and 12.2.2 to solve for $F_X(x;\, t)$. The cases of $\sin(\omega t) > 0$ and $\sin(\omega t) < 0$ are shown in Fig. 19.1.6.

Case 1: $\sin(\omega t) > 0$. There are no points of intersection on the a axis for $x \leq 0$, and hence $F_X(x; t) = 0$. For $0 < x \leq \sin(\omega t)$ we solve $x = a\sin(\omega t)$ and obtain $a = x/[\sin(\omega t)]$. The region I_a for which $a\sin(\omega t) \leq x$ is given by $I_a = \{0 < a \leq$

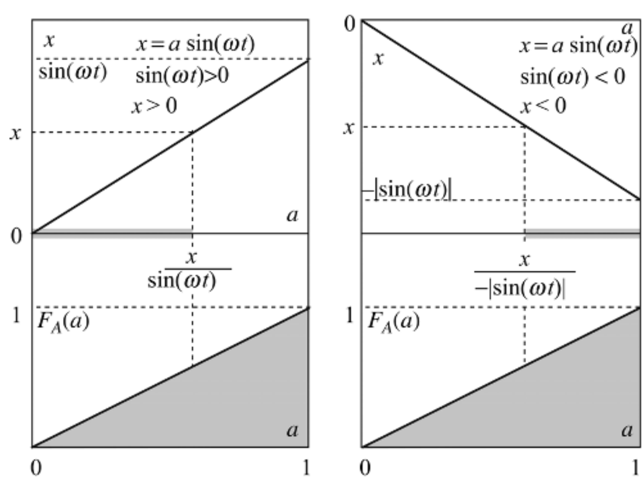


FIGURE 19.1.6

$x/[\sin(\omega t)]\}$ (Fig. 19.1.6). Thus

$$F_X(x;\, t) = F_A\left(\frac{x}{\sin(\omega t)}\right) - F_A(0) = F_A\left(\frac{x}{\sin(\omega t)}\right) = \frac{x}{\sin(\omega t)}$$

Finally for $x > \sin(\omega t)$, the region I_a for which $a\sin(\omega t) \le x$, is given by $I_a = \{0 < a \le 1\}$ and $F_X(x; t) = 1$. Thus, for $\sin(\omega t) > 0$, we have

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \le 0 \\ \dfrac{x}{\sin(\omega t)}, & 0 < x \le \sin(\omega t) \\ 1, & x > \sin(\omega t) \end{bmatrix}$$

Case 2: $\sin(\omega t) < 0$. The region I_a for which $x > 0$ is given by $I_a = \{0 < a \le 1\}$, and hence $F_X(x; t) = 1$. For $-|\sin(\omega t)| < x \le 0$, we solve $x = -a|\sin(\omega t)|$ and obtain $a = [x/(-|\sin(\omega t)|)]$. The region I_a for which $-a\,|\sin(\omega t)| \le x$ is given by

$$I_a = \left\{\frac{x}{-|\sin(\omega t)|} < a \le 1\right\}$$

(Fig. 19.1.6). Thus,

$$F_X(x;\, t) = F_A(1) - F_A\left(\frac{x}{-|\sin(\omega t)|}\right) = 1 - \frac{x}{-|\sin(\omega t)|}$$

Finally, for $x \le -|\sin(\omega t)|$, the region I_a for which $-a\,|\sin(\omega t)| \le x$ is given by $I_a = \{1 < a \le \infty\}$ and $F_X(x; t) = 0$. Thus, for $\sin(\omega t) < 0$, we have

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \le -|\sin(\omega t)| \\ 1 - \dfrac{x}{-|\sin(\omega t)|}, & -|\sin(\omega t)| < x \le 0 \\ 1, & x > 0 \end{bmatrix}$$

Case 3: $\sin(\omega t) = 0$. The region I_a for which $x > 0$ is given by $I_a = \{0 < a \le 1\}$ and $F_X(x; t) = 1$. For $x \le 0$, $I_a = \varnothing$ and $F_X(x; t) = 0$. Thus, for $\sin(\omega t) = 0$, we have

$$F_X(x;\, t) = \begin{bmatrix} 0, & x \le 0 \\ 1, & x > 0 \end{bmatrix}$$

Example 19.1.5 We shall now find the distribution and density functions along with the mean and variance for the random process of Example 19.1.2 and see how this process differs from the previous ones.

We are given that $X(t) = A\,\sin(\omega t + \Phi)$, where A is a constant and $f_\Phi(\phi) = 1/2\pi$ in the interval $(0, 2\pi)$ and we have to find $f_X(x; t)$. We will solve this problem by (1) finding the distribution $F_X(x; t)$ and differentiating it, and (2) by direct determination of the density function.

1. *Determination of Distribution Function $F_X(x; t)$.* The distribution function for Φ is given by $F_\Phi(\phi) = (\phi/2\pi), 0 < \phi \le 2\pi$. The two solutions for $x = A \sin(\omega t + \phi)$ are obtained from the two equations:

$$\sin(\omega t + \phi_1) = \frac{x}{A} \quad \text{and} \quad \sin(\pi - \omega t - \phi_2) = \frac{x}{A}$$

Hence the solutions are given by

$$\phi_1 = \sin^{-1}\left(\frac{x}{A}\right) - \omega t \quad \text{and} \quad \phi_2 = \pi - \sin^{-1}\left(\frac{x}{A}\right)$$

and are shown in Fig. 19.1.7.

For $x \le -A$, there are no points of intersection and hence $F_X(x;\, t) = 0$. For $-A < x \le A$, the set of points along the ϕ axis such that $A\,\sin(\omega t + \phi) \le x$ is $(0,\, \phi_1] \cup (\phi_2,\, 2\pi]$. Hence $F_X(x;\, t)$ is given by

$$\begin{aligned} F_X(x;t) &= F_\Phi(\phi_1) - F_\Phi(0) + F_\Phi(2\pi) - F_\Phi(\phi_2) \\ &= \frac{1}{2\pi}\left\{\sin^{-1}\left(\frac{x}{A}\right) - \omega t - 0 + 2\pi - \left[\pi - \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right]\right\} \\ &= \frac{1}{2\pi}\left\{2\sin^{-1}\left(\frac{x}{A}\right) + \pi\right\} = \frac{1}{\pi}\sin^{-1}\left(\frac{x}{A}\right) + \frac{1}{2}, \quad -A < x \le A \end{aligned}$$

Finally, for $x > A$, the entire curve $A\,\sin(\omega t + \phi)$ is below x, and $F_X(x;t) = 1$.

2. *Determination of Density Function $f_X(x;\, t)$*

(a) The two solutions to $x = A\,\sin(\omega t + \phi)$ have been found earlier.

(b) The absolute derivatives $|\partial x/\partial\phi||_{\phi_1}$ and $|\partial x/\partial\phi||_{\phi_2}$ are given by

$$\begin{aligned} \left|\frac{\partial x}{\partial \phi}\right|\bigg|_{\phi_1} &= A\cos(\omega t + \phi_1) = A\cos\left[\omega t + \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right] \\ &= A\cos\left[\sin^{-1}\left(\frac{x}{A}\right)\right] = A\frac{\sqrt{A^2 - x^2}}{A} = \sqrt{A^2 - x^2} \end{aligned}$$

and

$$\begin{aligned} \left|\frac{\partial x}{\partial \phi}\right|\bigg|_{\phi_2} &= A\cos(\omega t + \phi_2) = \left|A\cos\left[\omega t + \pi - \sin^{-1}\left(\frac{x}{A}\right) - \omega t\right]\right| \\ &= \left|A\cos\left[\pi - \sin^{-1}\left(\frac{x}{A}\right)\right]\right| = A\frac{\sqrt{A^2 - x^2}}{A} = \sqrt{A^2 - x^2} \end{aligned}$$

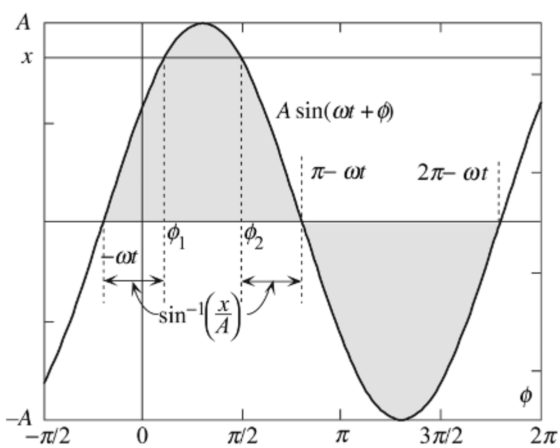


FIGURE 19.1.7

(c) With the two solutions for x, the density function $f_X(x;t)$ is given by Eq. (12.3.6):

$$f_X(x;\,t) = \frac{1}{\sqrt{A^2 - x^2}}\left(\frac{1}{2\pi} + \frac{1}{2\pi}\right) = \frac{1}{\pi\sqrt{A^2 - x^2}}, \quad -A < x \le A$$

Integration of $f_X(x;t)$ gives the distribution function $F_X(x;t)$

$$F_X(x;\,t) = \begin{bmatrix} 0, & x \le -A \\ \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{A}\right) + \dfrac{1}{2}, & -A < x \le A \\ 1, & x > A \end{bmatrix}$$

and these two solutions are exactly the same as before. For this random process, we find that the density and the distribution functions are both independent of time. For $A = 1$, they become

$$f_X(x;\,t) = \frac{1}{\pi\sqrt{1 - x^2}}, \qquad -1 < x \le 1$$

$$F_X(x;\,t) = \begin{bmatrix} 0, & x \le -1 \\ \dfrac{\sin^{-1}(x)}{\pi} + \dfrac{1}{2}, & -1 < x \le 1 \\ 1, & x > 1 \end{bmatrix}$$

and these functions are shown in Fig. 19.1.8. Since $f_X(x)$ has even symmetry, the mean value is 0 and the variance is obtained from

$$\sigma_X^2 = \int_{-1}^{1} \frac{x^2}{\pi\sqrt{1 - x^2}}\,dx = \frac{1}{2}$$

The value of $\sigma = \pm 1/\sqrt{2}$ is also shown in Fig. 19.1.8.

Later we will discuss random processes whose density and distribution functions are independent of time.

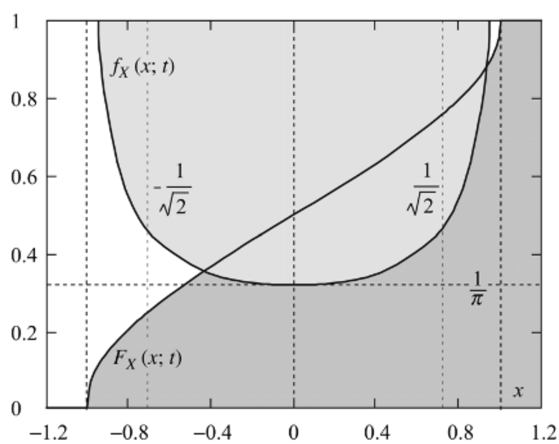


FIGURE 19.1.8

Higher-Order Distribution Functions

If t_1 and t_2 are different times, then $X_1 = X(t_1)$ and $X_2 = X(t_2)$ are two different random variables as shown in Fig. 19.1.9.

A second-order distribution function $F_X(x_1, x_2;\, t_1, t_2)$ for X_1 and X_2 can be defined as

$$F_X(x_1, x_2;\, t_1, t_2) = P[X(t_1) \le x_1,\ X(t_2) \le x_2] \tag{19.1.6}$$

and the second-order density function $f_X(x_1, x_2;\, t_1, t_2)$ as

$$f_X(x_1, x_2;\, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2;\, t_1, t_2) \tag{19.1.7}$$

Similarly, if $t_1, \ldots, t_n$ are different times, then an nth-order distribution function is defined as

$$F_X(x_1, \ldots, x_n;\, t_1, \ldots, t_n) = P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n] \tag{19.1.8}$$

and the nth-order density function as

$$f_X(x_1, \ldots, x_n;\, t_1, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_X(x_1, \ldots, x_n;\, t_1, \ldots, t_n) \tag{19.1.9}$$

A random variable is completely defined if its distribution function is known. Similarly, a random process $X(t)$ is completely defined if its nth-order distribution is known for all n. Since this is not feasible for all n, a random process in general cannot be completely defined. Hence, we usually restrict the definition to second-order distribution function, in which case it is called a *second-order process*.

In a similar manner, we can define a joint distribution between two different random processes $X(t)$ and $Y(t)$ (Fig. 19.1.10) as given below, where the joint second-order distribution function $F_{XY}(x_1, y_2;\, t_1, t_2)$ for $X(t_1)$ and $Y(t_2)$ is defined by

$$F_{XY}(x_1, y_2;\, t_1, t_2) = P[X(t_1) \le x_1,\ Y(t_2) \le y_2] \tag{19.1.10}$$

and the joint density function $f_{XY}(x_1, y_2;\, t_1, t_2)$ as

$$f_{XY}(x_1, y_2;\, t_1, t_2) = \frac{\partial^2}{\partial x_1\ \partial y_2} F_{XY}(x_1, y_2;\, t_1, t_2) \tag{19.1.11}$$

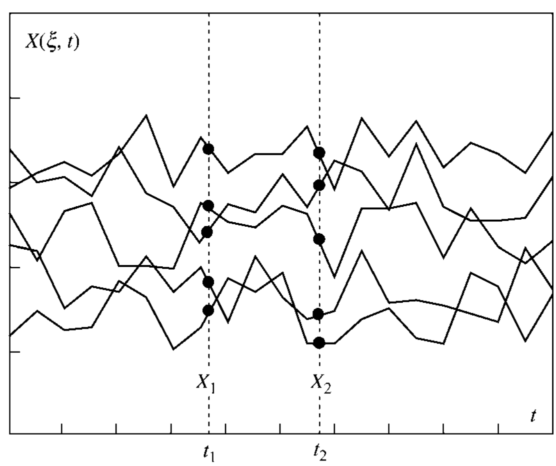


FIGURE 19.1.9

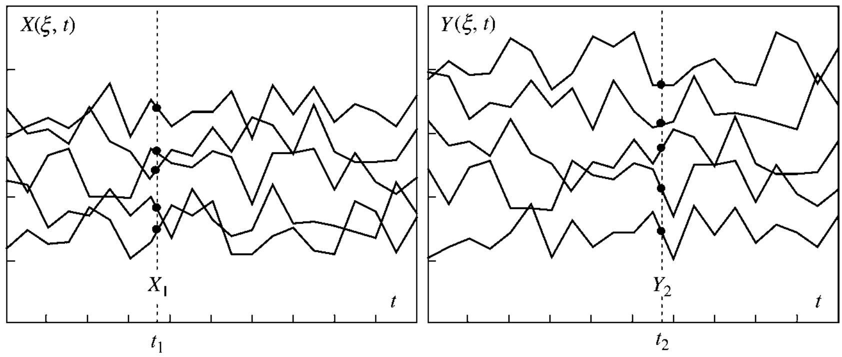


FIGURE 19.1.10

The joint nth-order distribution function $F_{XY}(x_1, y_2;\, t_1, t_2)$ for $X(t)$ and $Y(t)$ is defined by

$$F_{XY}(x_1, \ldots, x_n : y_1, \ldots, y_n;\, t_1, \ldots, t_n) \\ = P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n : Y(t_1) \le y_1, \ldots, Y(t_n) \le y_n] \qquad (19.1.12)$$

and the corresponding nth-order density function, by

$$f_{XY}(x_1, \ldots, x_n : y_1, \ldots, y_n;\, t_1, \ldots, t_n) \\ = \frac{\partial^{2n}}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_n} F_{XY}(x_1, \ldots, x_n : y_1, \ldots, y_n;\, t_1, \ldots, t_n) \qquad (19.1.13)$$

Second Order Moments

In Eqs. (19.1.3) and (19.1.4) defined mean and variance for a random process. The second moment of a random process has also been defined in Eq. (19.1.5). Since $X(t_1)$ and $X(t_2)$ are random variables, various types of joint moments can be defined.

Autocorrelation. The autocorrelation function (AC) $R_X(t_1, t_2)$ is defined as the expected value of the product $X(t_1)$ and $X(t_2)$:

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2;\, t_1, t_2)dx_1\, dx_2 \qquad (19.1.14)$$

By substituting $t_1 = t_2 = t$ in Eq. (19.1.14), we can obtain the second moment or the *average power* of the random process:

$$R_X(t) = E\,[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x;\, t)dx \qquad (19.1.15)$$

Autocovariance. The autocovariance function (ACF) $C_X(t_1, t_2)$ is defined as the covariance between $X(t_1)$ and $X(t_2)$:

$$C_X(t_1, t_2) = E\,[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X(t_1))(x_2 - \mu_X(t_2)) f_X(x_1, x_2;\, t_1, t_2)dx_2\, dx_1 \\ = E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2) \qquad (19.1.16)$$

Thus, from Eqs. (19.1.15) and (19.1.16) the interrelationships between AC and ACF are

$$\begin{aligned} C_X(t_1,t_2) &= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \\ R_X(t_1,t_2) &= C_X(t_1,t_2) + \mu_X(t_1)\mu_X(t_2) \end{aligned} \tag{19.1.17}$$

Normalized Autocovariance. The normalized autocovariance function (NACF) $\rho_X(t_1, t_2)$ is the ACF normalized by the variance and is defined by

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}} = \frac{C_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)} \tag{19.1.18}$$

The NACF finds wide applicability in many problems in random processes, particularly in time-series analysis.

The following definitions pertain to two different random processes $X(t)$ and $Y(t)$:

Cross-Correlation. The cross-correlation function (CC) $R_{XY}(t_1, t_2)$ is defined as the expected value of the product $X(t_1)$ and $Y(t_2)$:

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_{XY}(x_1,y_2;\, t_1,t_2)dy_2\, dx_1 \tag{19.1.19}$$

By substituting $t_1 = t_2 = t$ in Eq. (19.1.19), we can obtain the joint moment between the random processes $X(t)$ and $Y(t)$ as

$$R_{XY}(t) = E[X(t)Y(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y;\, t)dydx \tag{19.1.20}$$

Cross-Covariance. The cross-covariance function (CCF) $C_{XY}(t_1, t_2)$ is defined as the covariance between $X(t_1)$ and $Y(t_2)$:

$$\begin{aligned} C_{XY}(t_1,t_2) &= E[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X(t_1))(y_2 - \mu_Y(t_2)) f_{XY}(x_1,y_2;\, t_1,t_2)dy_2\, dx_1 \\ &= E[X(t_1)Y(t_2)] - \mu_X(t_1)\mu_Y(t_2) \end{aligned} \tag{19.1.21}$$

Thus, from Eqs. (19.1.19) and (19.1.21) the interrelationships between CC and CCF are

$$\begin{aligned} C_{XY}(t_1,t_2) &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) \\ R_{XY}(t_1,t_2) &= C_{XY}(t_1,t_2) + \mu_X(t_1)\mu_Y(t_2) \end{aligned} \tag{19.1.22}$$

Normalized Cross-Covariance. The normalized cross-covariance function (NCCF) $\rho_{XY}(t_1, t_2)$ is the CCF normalized by the variances and is defined by

$$\rho_{XY}(t_1,t_2) = \frac{C_{XY}(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_Y^2(t_2)}} = \frac{C_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)} \tag{19.1.23}$$

Some Properties of X(t) and Y(t)

Two random processes $X(t)$ and $Y(t)$ are *independent* if for all t_1 and t_2

$$F_{XY}(x,y;\, t_1,t_2) = F_X(x;\, t_1)F_Y(y;\, t_2) \tag{19.1.24}$$

They are *uncorrelated* if

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2) = 0$$

or

$$R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2) \quad \text{for all } t_1 \text{ and } t_2 \tag{19.1.25}$$

They are *orthogonal* if for all t_1 and t_2

$$R_{XY}(t_1, t_2) = 0 \tag{19.1.26}$$

Example 19.1.6 This example is slightly different from Example 19.1.4. A random process $X(t)$ is given by $X(t) = A\sin(\omega t + \phi)$ as shown in Fig. 19.1.11, where A is a uniformly distributed random variable with mean μ_A and variance σ_A^2. We will find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance of $X(t)$.

Mean:

$$E[X(t)] = \mu_X(t) = E[A\sin(\omega t + \phi)] = \mu_A \sin(\omega t + \phi)$$

Variance:

$$\begin{aligned}\text{var}[X(t)] = \sigma_X^2(t) &= E[A^2 \sin^2(\omega t + \phi)] - \mu_A^2 \sin^2(\omega t + \phi)\\ &= \{E[A^2] - \mu_A^2\} \sin^2(\omega t + \phi) = \sigma_A^2 \sin^2(\omega t + \phi)\end{aligned}$$

Autocorrelation. From Eq. (19.1.14) we have

$$\begin{aligned}R_X(t_1, t_2) = E[X(t_1)X(t_2)] &= E[A^2]\sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)\\ &= \frac{1}{2}E[A^2]\{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2\phi]\}\end{aligned}$$

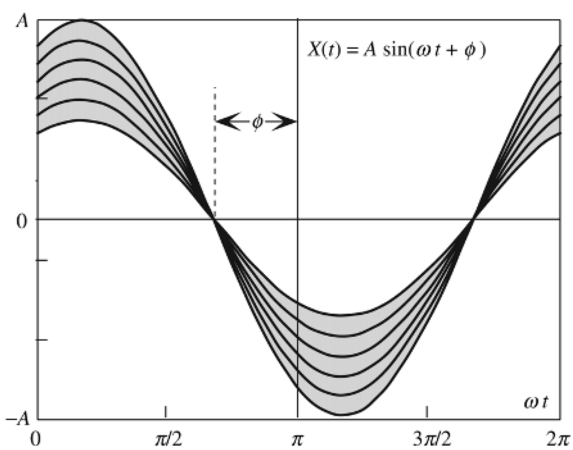


FIGURE 19.1.11

Autocovariance. From Eq. (19.1.16) we have

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\ &= E[A^2]\sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi) - \mu_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi) \\ &= \sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi) \\ &= \frac{1}{2}\sigma_A^2\{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2\phi]\} \end{aligned}$$

Normalized Autocovariance. From Eq. (19.1.18) we have

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{\sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)}{\sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)} = 1$$

Example 19.1.7 A random process $X(t)$ with k changes in a time interval t, and its probability mass function is given by $p(k; \lambda) = e^{-\lambda t}[(\lambda t)^k/k!]$. It is also known that the joint probability $P\{k_1$ changes in t_1, k_2 changes in $t_2\}$ is given by

$$\begin{aligned} &P\{k_1 \text{ changes in } t_1, k_2 \text{ changes in } t_2\} \\ &\quad = P\{k_1 \text{ changes in } t_1, (k_2 - k_1) \text{ changes in } (t_2 - t_1)\} \\ &\quad = P\{k_1 \text{ changes in } t_1\}P\{(k_2 - k_1) \text{ changes in } (t_2 - t_1)\} \\ &\quad = e^{-\lambda t_1}\frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda(t_2 - t_1)}\frac{[\lambda(t_2 - t_1)]^{(k_2 - k_1)}}{(k_2 - k_1)!} \end{aligned}$$

We have to find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance of $X(t)$:

Mean:

$$E[X(t)] = \mu_X(t) = \sum_{k=0}^{\infty} k e^{-\lambda t}\frac{(\lambda t)^k}{k!} = \lambda t$$

Variance:

$$\text{var}[X(t)] = \sigma_X^2(t) = \sum_{k=0}^{\infty} k^2 e^{-\lambda t}\frac{(\lambda t)^k}{k!} - (\lambda t)^2 = \lambda t$$

Autocorrelation. From Eq. (19.1.18) we have

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E\{X(t_1)[X(t_2) - X(t_1) + X(t_1)]\} \\ &= E[X^2(t_1)] + E[X(t_1)]E[X(t_2) - X(t_1)] \quad \text{(from condition given)} \\ &= (\lambda t_1)^2 + \lambda t_1 + \lambda t_1 \lambda(t_2 - t_1) \\ &= \lambda^2 t_1 t_2 + \lambda t_1 \text{ if } t_2 > t_1 \end{aligned}$$

and we have a similar result if $t_1 > t_2$:

$$R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda t_2 \quad \text{if} \quad t_1 > t_2$$

Combining these two results, we have

$$R_X(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Autocovariance. From Eq. (19.1.19) we have

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \\ &= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) - \lambda^2 t_1 t_2 = \lambda \min(t_1, t_2) \end{aligned}$$

Normalized Autocovariance. From Eq. (19.1.18) we have

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{\min(t_1, t_2)}{\lambda\sqrt{t_1 t_2}} = \frac{1}{\lambda}\min\left(\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_1}}\right)$$

Example 19.1.8 Two random processes $X(t)$ and $Y(t)$ are given by

$$X(t) = A\sin(\omega t + \phi_1); \quad Y(t) = B\sin(\omega t + \phi_2)$$

where A and B are two random variables with parameters $E[A] = \mu_A$, $E[B] = \mu_B$, $\text{var}[A] = \sigma_A^2$, $\text{var}[B] = \sigma_B^2$, $\text{cov}[AB] = \sigma_{AB}$, and correlation coefficient $\rho_{AB} = \sigma_{AB}/(\sigma_A\sigma_B)$. We have to find the means and variances of $X(t)$ and $Y(t)$ and their cross-correlation, cross-covariance, and normalized cross-covariance. The means and variances can be obtained directly from Example 19.1.5.

Means:

$$\begin{aligned} \mu_X(t) &= E[A\sin(\omega t + \phi_1)] = \mu_A \sin(\omega t + \phi_1) \\ \mu_Y(t) &= E[B\cos(\omega t + \phi_2)] = \mu_B \cos(\omega t + \phi_2) \end{aligned}$$

Variances:

$$\begin{aligned} \sigma_X^2(t) &= \sigma_A^2 \sin^2(\omega t + \phi_1) \\ \sigma_Y^2(t) &= \sigma_B^2 \cos^2(\omega t + \phi_2) \end{aligned}$$

Cross-Correlation:

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] = E[AB]\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2) \\ &= \frac{1}{2}E[AB]\{\sin[\omega(t_1 + t_2) + \phi_1 + \phi_2] + \sin[\omega(t_1 - t_2) + \phi_1 - \phi_2]\} \end{aligned}$$

Cross-Covariance:

$$\begin{aligned} C_{XY}(t_1, t_2) &= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2) \\ &= E[AB]\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2) - \mu_A\mu_B\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2) \\ &= \sigma_{AB}\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2) \\ &= \frac{1}{2}\sigma_{AB}\{\sin[\omega(t_1 + t_2) + \phi_1 + \phi_2] + \sin[\omega(t_1 - t_2) + \phi_1 - \phi_2]\} \end{aligned}$$

Normalized Cross-Covariance:

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sigma_X(t_1)\sigma_Y(t_2)} = \frac{\sigma_{AB}\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2)}{\sigma_A\sigma_B\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2)} = \frac{\sigma_{AB}}{\sigma_A\sigma_B} = \rho_{AB}$$

19.2 STATIONARY RANDOM PROCESSES

The distribution functions of all the random processes except that in Example 19.1.4 were dependent on time. However, many of the random processes such as Example 19.1.4 have the important property that their statistics do not change with time, which is an important step toward obtaining the statistics from a single sample function. The statistics in the time interval (t_1, t_2) is the same as in the time interval $(t_1 + \tau, t_2 + \tau)$. In other words, the probabilities of the samples of a random process $X(t)$ at times $t_1, \ldots, t_n$ will not differ from those at times $t_1 + \tau, \ldots, t_n + \tau$. This means that the joint distribution function of $X(t_1), \ldots, X(t_n)$ is the same as $X(t_1 + \tau), \ldots, X(t_n + \tau)$, or

$$F_X(x_1, \ldots, x_n;\, t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n;\, t_1 + \tau, \ldots, t_n + \tau) \tag{19.2.1}$$

and the corresponding density function may be written as

$$f_X(x_1, \ldots, x_n;\, t_1, \ldots, t_n) = f_X(x_1, \ldots, x_n;\, t_1 + \tau, \ldots, t_n + \tau) \tag{19.2.2}$$

Random processes with the property of Eq. (19.2.1) or (19.2.2) are called nth-order *stationary processes*. A *strict-sense* or *strongly* stationary process is a random process that satisfies Eqs. (19.2.1) and (19.2.2) for all n. Analogously, we can also define lower orders of stationarity.

A random process is *first-order stationary* if

$$\begin{aligned} F_X(x;\, t) &= F_X(x;\, t + \tau) = F_X(x) \\ f_X(x;\, t) &= f_X(x;\, t + \tau) = f_X(x) \end{aligned} \tag{19.2.3}$$

and the distribution and density functions are *independent* of time. The random process in Examples 19.1.2 and 19.1.5 is an example of a first-order stationary process.

A random process is *second-order stationary* if

$$\begin{aligned} F_X(x_1, x_2;\, t_1, t_2) &= F_X(x_1, x_2;\, t_1 + \tau,\, t_2 + \tau) = F_X(x_1, x_2;\, \tau) \\ f_X(x_1, x_2;\, t_1, t_2) &= f_X(x_1, x_2;\, t_1 + \tau,\, t_2 + \tau) = f_X(x_1, x_2;\, \tau) \end{aligned} \tag{19.2.4}$$

The distribution and density functions are dependent not on two time instants t_1 and t_2 but on the time difference $\tau = t_1 - t_2$ only. Second-order stationary processes are also called *wide-sense stationary* or *weakly stationary*. Hereafter, *stationary* means wide-sense stationary, and strict-sense stationary will be specifically mentioned.

In a similar manner, two processes $X(t)$ and $Y(t)$ are jointly stationary if for all n

$$\begin{aligned} &F_{XY}(x_1, \ldots, x_n;\, y_1, \ldots, y_n;\, t_1, \ldots, t_n) \\ &\quad = F_{XY}(x_1, \ldots, x_n;\, y_1, \ldots, y_n;\, t_1 + \tau, \ldots, t_n + \tau) \end{aligned} \tag{19.2.5}$$

or

$$\begin{aligned} &f_{XY}(x_1, \ldots, x_n;\, y_1, \ldots, y_n;\, t_1, \ldots, t_n) \\ &\quad = f_{XY}(x_1, \ldots, x_n;\, y_1, \ldots, y_n;\, t_1 + \tau, \ldots, t_n + \tau) \end{aligned} \tag{19.2.6}$$

and they are jointly wide-sense stationary if

$$\begin{aligned} &F_{XY}(x_1, y_2;\, t_1, t_2) \\ &\quad = F_{XY}(x_1, y_2;\, t_1 + \tau,\, t_2 + \tau) = F_{XY}(x_1, y_2;\, \tau) \end{aligned} \tag{19.2.7}$$

and the joint density function

$$f_{XY}(x_1, y_2;\ t_1, t_2) = f_{XY}(x_1, y_2;\ t_1 + \tau,\ t_2 + \tau) = f_{XY}(x_1, y_2;\ \tau) \qquad (19.2.8)$$

nth-order stationarity implies lower-order stationarities. Strict-sense stationarity implies wide-sense stationarity.

Similar to Eqs. (19.1.24)–(19.1.26), we can enumerate the following properties for stationary random processes $X(t)$ and $Y(t)$. Two random processes $X(t)$ and $Y(t)$ are *independent* if for all x and y

$$F_{XY}(x, y) = F_X(x)F_Y(y) \qquad (19.2.9)$$

They are *uncorrelated* if for all τ

$$C_{XY}(\tau) = R_{XY}(\tau) - \mu_X\mu_Y = 0$$

or

$$R_{XY}(\tau) = \mu_X\mu_Y \qquad (19.2.10)$$

They are *orthogonal* if for all τ

$$R_{XY}(\tau) = 0 \qquad (19.2.11)$$

Moments of Continuous-Time Stationary Processes

We can now define the various moments for a stationary random process $X(t)$.

Mean:

$$E[X(t)] = \mu_X = \int_{-\infty}^{\infty} xf(x)dx \qquad (19.2.12)$$

Variance:

$$E[X(t) - \mu_X]^2 = \sigma_X^2 = \int_{-\infty}^{\infty} (x - m_X)^2 f(x)dx = E[X^2(t)] - \mu_X^2 \qquad (19.2.13)$$

Autocorrelation:

$$R_X(\tau) = E[X(t)X(t + \tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f(x_1, x_2;\ \tau)dx_1\ dx_2 \qquad (19.2.14)$$

Autocovariance:

$$\begin{aligned} C_X(\tau) &= E\{[X(t) - \mu_X][X(t + \tau) - \mu_X]\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X)(x_2 - \mu_X) f(x_1, x_2;\ \tau)dx_1\ dx_2 \\ &= R_X(\tau) - \mu_X^2 \end{aligned} \qquad (19.2.15)$$

Normalized Autocovariance (NACF):

$$\rho_X(\tau) = \frac{C_X(\tau)}{\sigma_X^2} \qquad (19.2.16)$$

White-Noise Process

A zero mean stationary random process $X(t)$ whose autocovariance or autocorrelation is given by

$$C_X(\tau) = R_X(\tau) = \sigma_X^2 \delta(\tau) \tag{19.2.17}$$

where $\delta(\tau)$ is the Dirac delta function, is called a *white-noise process*. The energy of a white-noise process is infinite since $C_X(0) = R_X(0) = E[X^2(t)] = \infty$. Hence it is an idealization. White-noise processes find extensive use in modeling communication systems.

The cross-moments of two jointly stationary processes $X(t)$ and $Y(t)$ are defined below:

Cross-Correlation:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f(x_1, y_2; \tau) dy_2\, dx_1 \tag{19.2.18}$$

Cross-Covariance:

$$\begin{aligned} C_{XY}(\tau) &= E\{[X(t) - \mu_X][Y(t+\tau) - \mu_Y]\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X)(y_2 - \mu_Y) f(x_1, y_2; \tau) dy_2\, dx_1 \\ &= R_{XY}(\tau) - \mu_X \mu_Y \end{aligned} \tag{19.2.19}$$

Normalized Cross-Covariance:

$$\rho_{XY}(\tau) = \frac{C_{XY}(\tau)}{\sigma_X \sigma_Y} \tag{19.2.20}$$

A stationary random process $X(t)$ is passed through a linear system with impulse response $h(t)$. The input–output relationship will be given by the convolution integral:

$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha = \int_{-\infty}^{\infty} X(t)h(t-\alpha)d\alpha \tag{19.2.21}$$

The cross-correlation function $R_{XY}(\tau)$ between the input and the output can be found as follows:

$$E[X(t)Y(t+\tau)] = E\int_{-\infty}^{\infty} X(t)X(t+\tau-\alpha)h(\alpha)d\alpha \tag{19.2.22}$$

or

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_X(\tau - \alpha)h(\alpha)d\alpha \tag{19.2.23}$$

Since the cross-correlation function depends only on τ, the output $Y(t)$ will also be a stationary random process.

Properties of Correlation Functions of Stationary Processes

Autocorrelation Functions

1. $R_X(0) = E[X^2(t)] =$ average power ≥ 0
2. $R_X(\tau) = R_X(-\tau)$. Or, $R_X(\tau)$ is an even function.

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = R_X(-\tau) \tag{19.2.24}$$

3. $|R_X(\tau)| \le R_X(0)$. Or, the maximum value of $|R_X(\tau)|$ occurs at $\tau = 0$ and $|R_X(\tau)|$ for any τ cannot exceed the value $R_X(0)$. From

$$E[X(t) \pm X(t+\tau)]^2 \ge 0$$

we have

$$E[X^2(t)] + E[X^2(t+\tau)] \pm 2E[X(t)X(t+\tau)] \ge 0$$

or

$$R_X(0) \ge |R_X(\tau)| \tag{19.2.25}$$

4. If a constant $T > 0$ exists such that $R_X(T) = R_X(0)$, then the $R_X(\tau)$ is periodic and $X(t)$ is called a *periodic stationary process*. From Schwartz' inequality [Eq. (14.5.13)] we have

$$\{E[g(X)h(X)]\}^2 \le E[g^2(X)]E[h^2(X)]$$

Substituting $g(X) = X(t)$ and $h(X) = [X(t+\tau+T) - X(t+\tau)]$, we can write

$$\{E[X(t)[X(t+\tau+T) - X(t+\tau)]]\}^2 \le E[X^2(t)]E[X(t+\tau+T) - X(t+\tau)]^2$$

or

$$\{R_X(\tau+T) - R_X(\tau)\}^2 \le 2R_X(0)[R_X(0) - R_X(T)]$$

Hence, if $R_X(T) = R_X(0)$, then, since $\{R_X(\tau+T) - R_X(\tau)\}^2 \ge 0$, the result $R_X(\tau+T) = R_X(\tau)$ follows.

5. $\mathrm{E}[X(t+\phi)X(t+\tau+\phi) = R_X(\tau) = E[X(t)X(t+\tau)]$
$E[X(t+\phi)Y(t+\tau+\phi) = R_{XY}(\tau) = E[X(t)Y(t+\tau)]$

In the formulation of correlation functions the phase information is lost.

6. If $E[X(t)] = \mu_X$ and $Y(t) = a + X(t)$, where a is constant, then $E[Y(t)] = a + \mu_X$ and,

$$\begin{aligned} R_Y(\tau) &= E\{[a + X(t)][a + X(t+\tau)]\} \\ &= a^2 + aE[X(t+\tau)] + aE[X(t)] + E[X(t)X(t+\tau)] \\ &= a^2 + 2a\mu_X + R_X(\tau) = a^2 + 2a\mu_X + \mu_X^2 + C_X(\tau) \\ &= (a+\mu_X)^2 + C_X(\tau) \end{aligned} \tag{19.2.26}$$

If $E[X(t)] = 0$, then $E[Y(t)] = a$, and we can obtain the mean value of $Y(t)$ from a knowledge of its autocorrelation function $R_Y(\tau)$.

Cross-Correlation Functions

7. $R_{XY}(\tau) = R_{YX}(-\tau)$: This is not an even function. (19.2.27)

The result follows from the definition of cross-correlation:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[Y(t+\tau)X(t)] = R_{YX}(-\tau) \quad \text{but} \quad R_{XY}(0) = R_{YX}(0) \tag{19.2.28}$$

8. $R_{XY}^2(\tau) \leq R_X(0)R_Y(0)$: This result follows from Schwartz' inequality:

$$\{E[X(t)Y(t+\tau)]\}^2 \leq E[X^2(t)]E[Y^2(t+\tau)]$$

9. $2|R_{XY}(\tau)| \leq R_X(0) + R_Y(0)$: The result follows from

$$E[X(t) \pm Y(t+\tau)]^2 = R_X(0) + R_Y(0) \pm 2R_{XY}(\tau) \geq 0 \tag{19.2.29}$$

10. If $Z(t) = X(t) + Y(t)$, then

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] = E\{[X(t) + Y(t)][X(t+\tau) + Y(t+\tau)]\} \\ &= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau) \end{aligned} \tag{19.2.30}$$

and if $X(t)$ and $Y(t)$ are orthogonal, then $R_Z(\tau) = R_X(\tau) + R_Y(\tau)$.

11. If $\dot{X}(t)$ is the derivative of $X(t)$, then the cross-correlation between $X(t)$ and $\dot{X}(t)$ is

$$R_{X\dot{X}}(\tau) = \frac{dR_X(\tau)}{d\tau} \tag{19.2.31a}$$

This result can be shown from the formal definition of the derivative of $X(t)$. Substituting for $\dot{X} = \lim_{\varepsilon\to 0}\{[X(t+\varepsilon) - X(t)]/\varepsilon\}$, we obtain

$$\begin{aligned} R_{X\dot{X}}(\tau) &= E[X(t)X(t+\tau)] \\ &= \lim_{\varepsilon\to 0} E\left\{X(t)\left[\frac{X(t+\tau+\varepsilon) - X(t+\tau)}{\varepsilon}\right]\right\} \\ &= \lim_{\varepsilon\to 0} E\left\{\frac{X(t)X(t+\tau+\varepsilon) - X(t)X(t+\tau)}{\varepsilon}\right\} \\ &= \lim_{\varepsilon\to 0} \frac{R_X(\tau+\varepsilon) - R_X(\tau)}{\varepsilon} = \frac{dR_X(\tau)}{d\tau} \end{aligned}$$

12. The autocorrelation of $\dot{X}(t)$ is

$$R_{\dot{X}}(\tau) = -\frac{d^2R_X(\tau)}{d\tau^2} \tag{19.2.31b}$$

This result can be shown following a procedure similar to that described as in (11) above:

$$\begin{aligned}
R_{\dot{X}}(\tau) &= E[\dot{X}(t)\dot{X}(t+\tau)] \\
&= \lim_{\varepsilon\to 0} E\left[X(t)\frac{X(t+\tau+\varepsilon)-X(t+\tau)}{\varepsilon}\right] \\
&= \lim_{\delta\to 0}\lim_{\varepsilon\to 0} E\left[\left(\frac{X(t+\delta)-X(t)}{\delta}\right)\left(\frac{X(t+\tau+\varepsilon)-X(t+\tau)}{\varepsilon}\right)\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\lim_{\varepsilon\to 0} E\begin{bmatrix} X(t+\delta)\dfrac{X(t+\tau+\varepsilon)-X(t+\tau)}{\varepsilon} \\ -X(t)\dfrac{X(t+\tau+\varepsilon)-X(t+\tau)}{\varepsilon} \end{bmatrix} \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\lim_{\varepsilon\to 0}\left[\frac{R_X(\tau-\delta+\varepsilon)-R_X(\tau-\delta)}{\varepsilon}-\frac{R_X(\tau+\varepsilon)-R_X(\tau)}{\varepsilon}\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left[\frac{dR_X(\tau-\delta)}{d\tau}-\frac{dR_X(\tau)}{d\tau}\right] = -\frac{d^2R_X(\tau)}{d\tau^2}
\end{aligned}$$

It can be shown that if a random process is wide-sense stationary, then it is necessary and sufficient that the following two conditions be satisfied:

1. The expected value is a constant, $E[X(t)] = \mu_X$.
2. The autocorrelation function R_X is a function of the time difference $t_2 - t_1 = \tau$ and not individual times, $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$.

We will now give several examples to establish conditions for stationarity. The first few examples will be running examples that become progressively more difficult.

Example 19.2.1 We will revisit Example 19.1.5 and find the conditions necessary for the random process $X(t) = A\sin(\omega t + \Phi)$ to be stationary where A,ω are constants and Φ is a random variable:

1. *Mean*:

$$E[X(t)] = AE[\sin(\omega t + \Phi)] = A\int_{-a}^{b}\sin(\omega t + \phi)f_\Phi(\phi)d\phi$$

One of the ways the integral will be independent of t is for Φ to be uniformly distributed in $(0,2\pi)$, in which case we have

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(\omega t + \phi)d\phi = \frac{-1}{2\pi}\cos(\omega t + \phi)\Big|_0^{2\pi} = 0$$

and $E[X(t)] = 0$.

2. *Autocorrelation*:

$$E[X(t_1)X(t_2)] = A^2E[\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)]$$
$$= A^2E\{\cos[\omega(t_2 - t_1)] - \cos[\omega(t_2 + t_1) + 2\Phi]\}$$
$$= A^2\cos[\omega(t_2 - t_1)] - A^2E\{\cos[\omega(t_2 + t_1) + 2\Phi]\}$$

$$E\{\cos[\omega(t_2 + t_1) + 2\Phi]\} = \frac{1}{2\pi}\int_0^{2\pi}\cos[\omega(t_2 + t_1) + 2\phi]\,d\phi$$
$$= \frac{1}{2\pi}\frac{1}{2}\sin[\omega(t_2 + t_1) + 2\phi]\Big|_0^{2\pi} = 0$$

Hence, $R_X(t_1, t_2) = A^2\cos[\omega(t_2 - t_1)] = A^2\cos[\omega\tau]$, and $X(t)$ is stationary if Φ is uniformly distributed in $(0,2\pi)$. Since $R_X(\tau)$ is periodic, $X(t)$ is a periodic stationary process.

Example 19.2.2 We will modify Example 19.2.1 with both A and Φ as random variables with density functions $f_A(a)$ and $f_\Phi(\phi)$. We will now find the conditions under which $X(t) = A\sin(\omega t + \Phi)$ will be stationary.

1. *Mean*:

$$E[X(t)] = E[A\sin(\omega t + \Phi)] = \int\int a\sin(\omega t + \phi)f_{A\Phi}(a, \phi)d\phi\, da$$

The first condition for the double integral to be independent of t is for A and Φ to be statistically independent, in which case we have

$$E[A\sin(\omega t + \Phi)] = \int\int a\sin(\omega t + \phi)f_A(a)f_\Phi d\phi\, da$$

and the second condition is for Φ to be uniformly distributed in $(0,2\pi)$, in which case we have $(1/2\pi)\int_0^{2\pi}\sin(\omega t + \phi)\,d\phi = 0$ and $E[X(t)] = 0$.

2. *Autocorrelation*:

$$E[X(t_1)X(t_2)] = E[A^2\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)]$$

Since A and Φ are independent, we have

$$E[X(t_1)X(t_2)] = E[A^2]E\{\cos[\omega(t_2 - t_1) - \cos[\omega(t_2 + t_1) + 2\Phi]\}$$
$$= E[A^2]\cos[\omega(t_2 - t_1)] - E[A^2]E\{\cos[\omega(t_2 + t_1) + 2\Phi\}$$

and from the previous example $E\{\cos[\omega(t_2 + t_1) + 2\Phi]\} = 0$. Hence, $R_X(t_1, t_2) = E[A^2]\cos[\omega(t_2 - t_1)] = E[A^2]\cos[\omega\tau]$ and $X(t)$ is stationary if A and Φ are independent and if Φ is uniformly distributed in $(0,2\pi)$. Since $R_X(\tau)$ is periodic, $X(t)$ a periodic stationary process.

Example 19.2.3 We will now examine the conditions for stationarity for $X(t) = A\sin(\Omega t + \Phi)$ when A, Ω and Φ are all random variables with density functions $f_A(a)$, $f_\Omega(\omega)$, and $f_\Phi(\phi)$ respectively.

1. *Mean*:

$$E[X(t)] = E[A\sin(\Omega t + \Phi)] = \iiint a\sin(\omega t + \phi) f_{A\Omega\Phi}(a,\omega,\phi) d\phi\, d\omega\, da$$

This triple integral is a difficult one to evaluate. Hence we resort to conditional expectations by fixing the variable $\Omega = \omega$. Under this condition, $E[X(t)] = \int E[X(t)|\Omega = \omega]\ f_\Omega(\omega)d\omega$ and from the previous example, if A and Φ are independent and Φ is uniformly distributed in $(0, 2\pi)$, we have $E[X(t)|\Omega = \omega] = 0$, and

$$\int E[X(t)|\Omega = \omega] f_\Omega(\omega)d\omega = \int E[A\sin(\omega t + \Phi)] f_\Omega(\omega)d\omega = 0$$

2. *Autocorrelation*:

$$\begin{aligned} R_X(t_1, t_2|\Omega = \omega) &= E[X(t_1|\Omega = \omega)X(t_2|\Omega = \omega)] \\ &= E[A^2\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)] \end{aligned}$$

and from the previous example

$$E[A^2\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)] = \frac{1}{2}E[A^2]\cos[\omega\,\tau]$$

where $\tau = t_2 - t_1$. Hence

$$R_X(\tau) = \int R_X(\tau|\Omega = \omega) f_\Omega(\omega)d\omega = \int \frac{1}{2}E[A^2]\cos[\omega\tau] f_\Omega(\omega)d\omega$$

If Ω is uniformly distributed in $(0,\pi)$, then

$$R_X(\tau) = \frac{1}{\pi}\int_0^\pi \frac{1}{2}E[A^2]\cos[\omega\tau]d\omega = \frac{E[A^2]}{2\pi}\frac{\sin(\pi)}{\tau} = \frac{E[A^2]}{2}\mathrm{Sa}(\pi)$$

Example 19.2.4 In this example $X(t) = A\cos(\omega t) + B\sin(\omega t)$, where A and B are random variables with density functions $f_A(a)$ and $f_B(b)$. We have to find the conditions under which $X(t)$ will be stationary:

1. *Mean*:

$$E[X(t)] = E[A\cos(\omega t) + B\sin(\omega t)] = \cos(\omega t)E[A] + \sin(\omega t)E[B]$$

If $E[X(t)]$ is to be independent of t, then $E[A] = E[B] = 0$, in which case $E[X(t)] = 0$.

2. *Autocorrelation*:

$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E\{[A\cos(\omega t_1) \\ &\quad + B\sin(\omega t_1)][A\cos(\omega t_2) + B\sin(\omega t_2)]\} \\ &= E\{A^2\cos(\omega t_1)\cos(\omega t_2) + B^2\sin(\omega t_1)\sin(\omega t_2) \\ &\quad + AB[\sin(\omega t_1)\cos(\omega t_2) + \cos(\omega t_1)\sin(\omega t_2)]\} \\ &= \frac{1}{2}E[A^2][\cos(\omega(t_2 - t_1)) + \cos(\omega(t_2 + t_1))] \\ &\quad + \frac{1}{2}E[B^2][\cos(\omega(t_2 - t_1)) - \cos(\omega(t_2 + t_1))] \\ &\quad + E[AB]\sin(\omega(t_2 + t_1)) \end{aligned}$$

The conditions under which this equation will be dependent only on $(t_2 - t_1)$ are $E[A^2] = E[B^2]$ and $E[AB] = 0$. In this case

$$R_X(t_1,t_2) = E[A^2][\cos(\omega(t_2 - t_1))] = E[A^2][\cos(\omega\tau)]$$

We can now summarize the conditions for stationarity:

(a) $E[A] = E[B] = 0$

(b) $E[A^2] = E[B^2]$

(c) $E[AB] = 0$

Example 19.2.5 A die is tossed, and corresponding to the dots $S = \{1,2,3,4,5,6\}$, a random process $X(t)$ is formed with the following time functions as shown in Fig. 19.2.1:

$$X(2;t) = 3,\ X(4;t) = (2 - t),\ X(6;t) = (1 + t)$$
$$X(1;t) = -3,\ X(3;t) = -(2 - t),\ X(5;t) = -(1 + t)$$

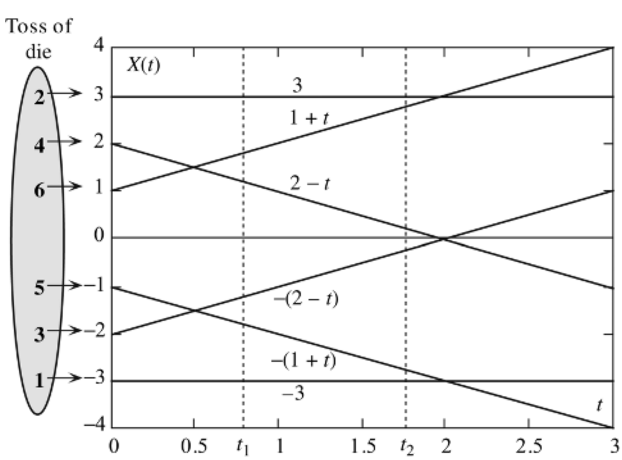


FIGURE 19.2.1

We have to find $\mu_X(t)$, $\sigma_X^2(t)$, $R_X(t_1,t_2)$, $C_X(t_1,t_2)$, and $\rho_X(t_1,t_2)$ and check whether $X(t)$ is stationary:

Mean:

$$\mu_X(t) = \frac{1}{6}\sum_{i=1}^{6} X_i(t) = 3 - 3 + (2-t) - (2-t) + (1+t) - (1+t) = 0$$

The mean value is a constant.

Variance:

$$\sigma_X^2(t) = \frac{1}{6}\sum_{i=1}^{6} X_i^2(t) = \frac{1}{3}\cdot[3^2 + (2-t)^2 + (1+t)^2] = \frac{2}{3}[t^2 - t + 7]$$

Autocorrelation:

$$R_X(t_1,t_2) = \frac{1}{3}[9 + (1+t_1)(1+t_2) + (2-t_1)(2-t_2)] = \frac{1}{3}[14 - t_2 - t_1 + 2t_1t_2]$$

Autocovariance. Since the mean value is zero, $C_X(t_1,t_2) = R_X(t_1,t_2)$.

Normalized Autocovariance:

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1)C_X(t_2)}} = \frac{[14 - t_2 - t_1 + 2t_1t_2]}{\sqrt{(t_1^2 - t_1 + 7)(t_2^2 - t_2 + 7)}}$$

This process is not stationary.

Example 19.2.6 (Random Binary Wave) A sample function of a random binary wave $X(t)$ consisting of independent rectangular pulses $p(t)$, each of which is of duration T, is shown in Fig. 19.2.2. The height H of the pulses is a random variable with constant amplitudes, which are equally likely to be $\pm A$. The time of occurrence of $X(t)$ after $t = 0$ is another random variable Z, which is uniformly distributed in $(0,T)$. We have to find the mean, variance, autocorrelation, autocovariance, and the normalized autocovariance of $X(t)$.

The random process $X(t)$ is given by

$$X(t) = \sum_{k=-\infty}^{\infty} Hp(t - kT - Z)$$

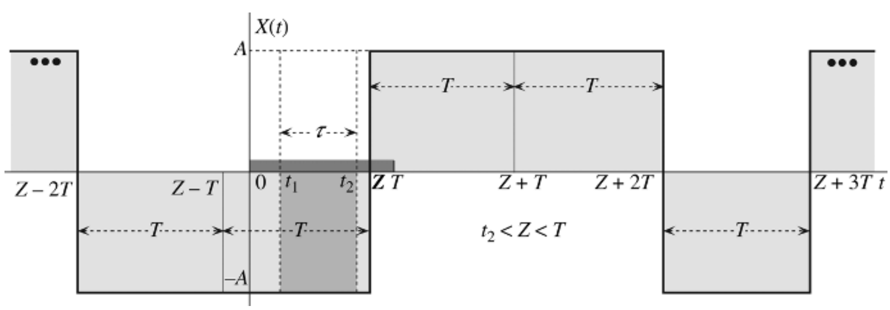


FIGURE 19.2.2

The probabilities of the random variable H are $P(H = A) = \frac{1}{2}$ and $P(H = -A) = \frac{1}{2}$. Since Z is uniformly distributed in $(0,T)$, the following probabilities can be formulated for $0 < t_1 < t_2 < T$:

$$P(t_1 < Z < t_2) = \frac{t_2 - t_1}{T}, \quad P(0 < Z < t_1) = \frac{t_1}{T}, \quad P(t_2 < Z < T) = 1 - \frac{t_2}{T}$$

Mean. Since $X(t)$ assumes only $+A$ or $-A$ with equal probability,

$$E[X(t)] = \left[\frac{1}{2} \cdot A + \frac{1}{2} \cdot (-A)\right] = 0$$

Variance. The variance is given by

$$\text{var}[X(t)] = \sigma_X^2 = E[X^2(t)] = \left[\frac{1}{2} \cdot A^2 + \frac{1}{2} \cdot (-A)^2\right] = A^2$$

Autocorrelation. Determining $R_X(t_1, t_2)$ is a bit more involved. The product $X(t_1)X(t_2)$ in various intervals of time (t_1, t_2) can be found:

$$t_1 < Z < t_2 \leq T$$

Here t_1 and t_2 lie in adjacent pulse intervals as shown in Fig. 19.2.3. In Fig. 19.2.3a $X(t_1) = -A$ and $X(t_2) = +A$ and $X(t_1)X(t_2) = -A^2$. In Fig. 19.2.3b $X(t_1) = +A$ and $X(t_2) = +A$ and $X(t_1)X(t_2) = A^2$. The values $-A^2$ and A^2 will occur with equal probability:

$0 < Z < t_1 \leq T$: In this case, t_1 and t_2 lie in the same pulse interval as shown in Fig. 19.2.3c. Here, $X(t_1) = +$A and $X(t_2) = +A$ and $X(t_1)X(t_2) = A^2$.

$t_2 < Z < T$: Here also t_1 and t_2 lie in the same pulse interval as shown in Fig. 19.2.3d. However, $X(t_1) = -A$ and $X(t_2) = -A$ and $X(t_1)X(t_2) = A^2$.

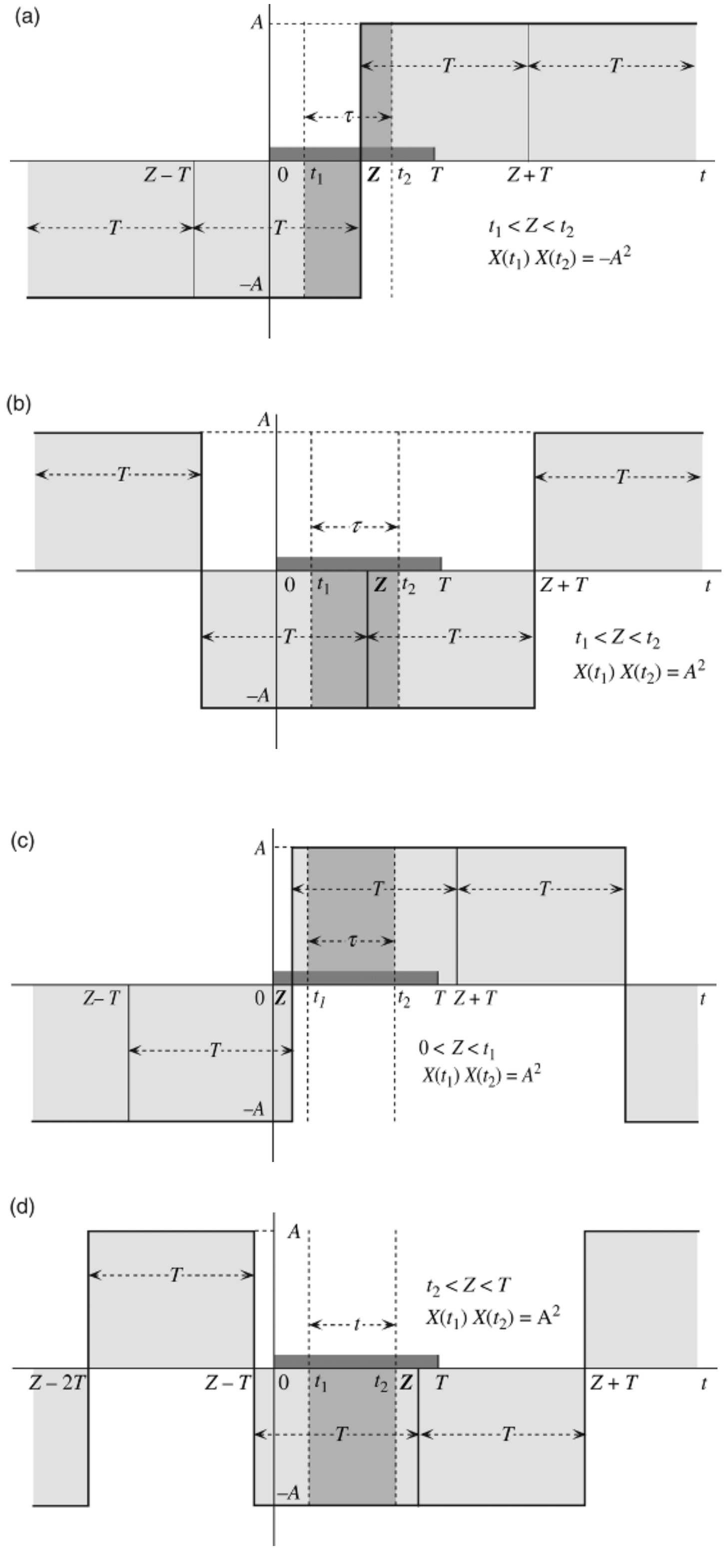
$(t_1 - t_2) > T$: In this case t_1 and t_2 lie in different pulse intervals and $X(t_1)X(t_2) = \pm A^2$.

We can now find the autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$. For $(t_1 - t_2) > T$, $X(t_1)$ and $X(t_2)$ are in different pulse intervals and invoking the independence of the pulses $E[X(t_1)X(t_2)] = E[X(t_1)]E[X(t_2)] = 0$ since $E[X(t)] = 0$. For $(t_1 - t_2) \leq T$, we have the following, using conditional expectations:

$$\begin{aligned}E[X(t_1)X(t_2)] = {} & E[X(t_1)X(t_2)|t_1 < Z < t_2]P(t_1 < Z < t_2) \\ & + E[X(t_1)X(t_2)|0 < Z < t_1]P(0 < Z < t_1) \\ & + E[X(t_1)X(t_2)|t_1 < Z < T]P(t_1 < Z < T)\end{aligned}$$

The conditional expectations can be determined using the probabilities and the products $X(t_1)X(t_2)$ found earlier for the various intervals. Since A^2 and $-A^2$ occur with equal probability in the interval $t_1 < Z < t_2 \leq T$, the first conditional expectation term becomes

$$E[X(t_1)X(t_2)|t_1 < Z < t_2]P(t_1 < Z < t_2) = A^2 \cdot \frac{1}{2} \cdot \frac{t_2 - t_1}{T} - A^2 \cdot \frac{1}{2} \cdot \frac{t_2 - t_1}{T} = 0$$


(a)
A
T
T
τ
Z − T
0
t1
Z
t2
T
Z + T
t
T
T
t1 < Z < t2
X(t1) X(t2) = −A2
−A
(b)
A
T
T
τ
0
t1
Z
t2
T
Z + T
t
T
T
t1 < Z < t2
X(t1) X(t2) = A2
−A
(c)
A
T
T
τ
Z− T
0
Z
t1
t2
T
Z + T
t
T
0 < Z < t1
X(t1) X(t2) = A2
−A
(d)
A
T
t2 < Z < T
X(t1) X(t2) = A2
t
Z − 2T
Z − T
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t1
t2
Z
T
Z + T
t
T
T
−A

FIGURE 19.2.3

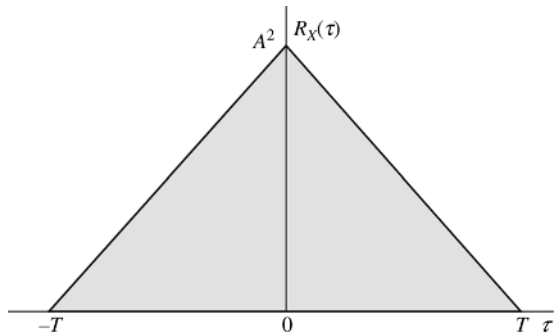

FIGURE 19.2.4

Hence

$$R_X(t_1, t_2) = \begin{cases} A^2\left(\frac{t_1}{T} + 1 - \frac{t_2}{T}\right) = A^2\left(1 - \frac{t_2 - t_1}{T}\right), & (t_2 - t_1) \le T \\ 0, & (t_2 - t_1) > T \end{cases}$$

This equation was derived under the assumption $t_1 < t_2$. For arbitrary values of t_1 and t_2 with $\tau = (t_2 - t_1)$, the autocorrelation function $R_X(t_1, t_2)$ is given by

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation function for the process $X(t)$ is shown in Fig. 19.2.4.

Autocovariance. Since the mean value is 0, $C_X(\tau) = R_X(\tau)$.

Normalized Autocovariance. The NACF is given by

$$\rho_X(\tau) = \frac{C_X(\tau)}{\sigma_X^2} = \begin{cases} 1 - \frac{|\tau|}{T}, & |\tau| < T \\ 0, & \text{otherwise} \end{cases}$$

Example 19.2.7 (Random Telegraph Wave) This example is a little different from the previous one. A sample function of a random telegraph wave is shown in Fig. 19.2.5. The wave assumes either of the values 1 or 0 at any instant of time.

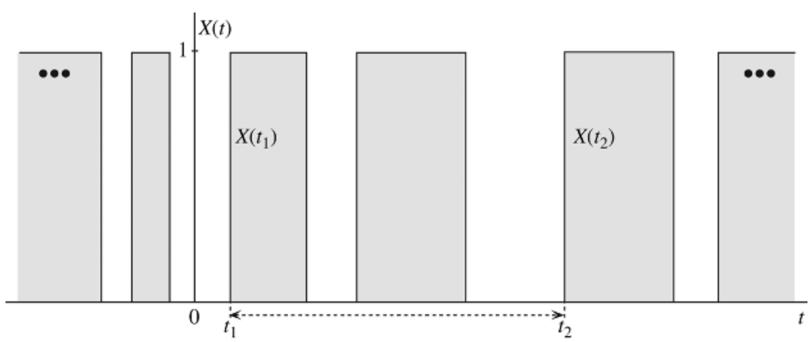

FIGURE 19.2.5

The probability of k changes from 0 to 1 in a time interval t is Poisson-distributed with probability mass function $p(k; \lambda)$ given by

$$p(k; \lambda) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0$$

where λ is the average number of changes per unit time. We have to find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance.

Mean. Since $X(t)$ assumes only 1 or 0 with equal probability, we obtain

$$E[X(t)] = \mu_X = 1 \cdot P(x=1) + 0 \cdot P(x=0) = \frac{1}{2}$$

Variance. The variance is given by

$$\text{var}[X(t)] = \sigma_X^2 = E[X^2(t)] - \mu_X^2 = \left[\frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 0^2\right] - \frac{1}{4} = \frac{1}{4}$$

Autocorrelation. The autocorrelation function can be given in terms of the joint density functions with $t_2 > t_1$:

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1) = 0 \cdot X(t_2) = 0] + E[X(t_1) = 0 \cdot X(t_2) = 1] \\ &\quad + E[X(t_1) = 1 \cdot X(t_2) = 0] + E[X(t_1) = 1 \cdot X(t_2) = 1] \\ &= (0 \cdot 0)P[X(t_1) = 0 \cdot X(t_2) = 0] + (0.1)P[X(t_1) = 0 \cdot X(t_2) = 1] \\ &\quad + (1 \cdot 0)P[X(t_1) = 1 \cdot X(t_2) = 0] + (1.1)P[X(t_1) = 1 \cdot X(t_2) = 1] \end{aligned}$$

or

$$R_X(t_1, t_2) = P[X(t_1) = 1 \cdot X(t_2) = 1]$$

Expressing the joint probability in terms of conditional probabilities, we have

$$R_X(t_1, t_2) = P[X(t_2) = 1 \mid X(t_1) = 1]P[X(t_1) = 1]$$

The conditional probability in this equation is the probability of even number of changes and using the Poisson distribution:

$$\begin{aligned} R_X(t_1, t_2) &= \frac{1}{2} \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{[\lambda(t_2 - t_1)]^k}{k!} e^{-\lambda(t_2 - t_1)} \\ &= \frac{e^{-\lambda(t_2-t_1)}}{2} \left\{ \frac{1}{2} \sum_{k=0}^{\infty} \frac{[\lambda(t_2 - t_1)]^k}{k!} + \sum_{k=0}^{\infty} \frac{-[\lambda(t_2 - t_1)]^k}{k!} \right\} \\ &= \frac{e^{-\lambda(t_2-t_1)}}{2} \left\{ \frac{e^{\lambda(t_2-t_1)} + e^{-\lambda(t_2-t_1)}}{2} \right\} \\ &= \frac{1}{4}\{1 + e^{-2\lambda(t_2-t_1)}\}, \quad t_2 > t_1 \end{aligned}$$

A similar equation holds good for $t_2 < t_1$. Hence, substituting $|t_2 - t_1| = |\tau|$ in the equation above, $R_X(\tau)$ can be given by

$$R_X(\tau) = \frac{1}{4}\{1 + e^{-2\lambda|\tau|}\}$$

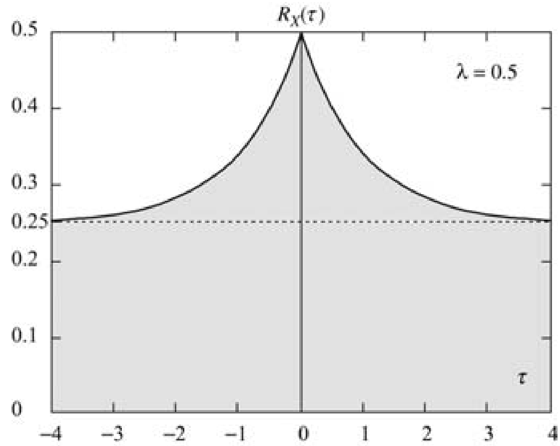

FIGURE 19.2.6

The autocorrelation function for the process $X(t)$ is shown in Fig. 19.2.6 for $\lambda = 0.5$.

Autocovariance:

$$C_X(\tau) = R_X(\tau) - \mu_X^2 = \frac{e^{-2\lambda|\tau|}}{4}$$

Normalized Autocovariance:

$$\rho_X(\tau) = \frac{C_X(\tau)}{\sigma_X^2} = e^{-2\lambda|\tau|}$$

Example 19.2.8 (Modulation) A random process $Y(t)$ is given by $Y(t) = X(t)\cos(\omega t + \Phi)$, where $X(t)$, a zero mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = 2e^{-2\lambda|\tau|}$ is modulating the carrier $\cos(\omega t + \Phi)$. The random variable Φ is uniformly distributed in the interval $(0,2\pi)$, and is independent of $X(t)$. We have to find the mean, variance, and autocorrelation of $Y(t)$:

Mean. The independence of $X(t)$ and Φ allows us to write

$$E[Y(t)] = E[X(t)]E[\cos(\omega t + \Phi)]$$

and with $E[X(t)] = 0$ and $E[\cos(\omega t + \Phi)] = 0$ from Example 19.2.1, $E[Y(t)] = 0$.

Variance. Since $X(t)$ and Φ are independent, the variance can be given by

$$\sigma_Y^2 = E[Y^2(t)] = E[X^2(t)\cos^2(\omega t + \Phi)] = \sigma_X^2 E[\cos^2(\omega t + \Phi)]$$

However

$$E[\cos^2(\omega t + \Phi)] = \frac{1}{2}E[1 + \cos(2\omega t + 2\Phi)] = \frac{1}{2} \quad \text{and} \quad \sigma_X^2 = C_X(0) = R_X(0) = 2$$

and hence $\sigma_Y^2 = \sigma_X^2/2 = 1$.

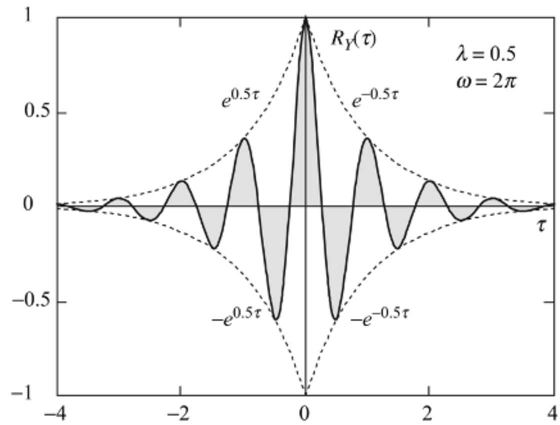

FIGURE 19.2.7

Autocorrelation:

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[X(t)\cos(\omega t + \Phi)X(t+\tau)\cos(\omega t + \omega\tau + \Phi)] \\ &= R_X(\tau)\frac{1}{2}E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\Phi)] \\ &= \frac{R_X(\tau)}{2}\cos(\omega\tau) + \frac{R_X(\tau)}{2}E[\cos(2\omega t + \omega\tau + 2\Phi)] \end{aligned}$$

From Example 19.2.1 $E[\cos(2\omega t + \omega\tau + 2\Phi)] = 0$, and hence

$$R_Y(\tau) = \frac{R_X(\tau)}{2}\cos(\omega\tau) = e^{-2\lambda|\tau|}\cos(\omega\tau)$$

A graph of $R_Y(\tau)$ is shown in Fig. 19.2.7 with $\lambda = 0.5$ and $\omega = 2\pi$.

Example 19.2.9 (Cross-Correlation) Two random processes $X(t)$ and $Y(t)$ are given by $X(t) = A\cos(\omega t) + B\sin(\omega t)$ and $Y(t) = -A\sin(\omega t) + B\cos(\omega t)$, where A and B are random variables with density functions $f_A(a)$ and $f_B(b)$. We have to find the cross-correlation, cross-covariance, and the normalized cross-covariance between $X(t)$ and $Y(t)$.

From Example 19.2.4 the processes $X(t)$ and $Y(t)$ are stationary if $E[A] = E[B] = 0$, $E[A^2] = E[B^2]$, and $E[AB] = 0$. Hence the mean values $\mu_X = \mu_Y = 0$. The variances of these processes are $E[A^2] = E[B^2] = \sigma^2$.

Cross-Correlation. The cross-correlation function $R_{XY}(t_1, t_2)$ can be written as

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E\{[A\cos(\omega t_1) + B\sin(\omega t_1)][-A\sin(\omega t_2) + B\cos(\omega t_2)]\} \\ &= E\{-A^2\cos(\omega t_1)\sin(\omega t_2) + B^2\sin(\omega t_1)\cos(\omega t_2) \\ &\quad + AB[\cos(\omega t_1)\cos(\omega t_2) - \sin(\omega t_1)\sin(\omega t_2)]\} \end{aligned}$$

Substituting the values $E[A^2] = E[B^2] = \sigma^2$ and $E[AB]=0$ in this equation, we obtain

$$R_{XY}(t_1,t_2) = \sigma^2 \sin[\omega(t_1 - t_2)] = -\sigma^2 \sin[\omega\tau]$$

where we have substituted $\tau = (t_2 - t_1)$. The term $R_{XY}(\tau)$ is shown in Fig. 19.2.8 for $\sigma^2=4$ and $\omega = 2\pi$.

Here $R_{XY}(\tau)$ is an odd function, unlike the autocorrelation function. Since $R_{XY}(\tau)=0$ for $\tau = 0$, we conclude that $X(t)$ and $Y(t)$ are orthogonal.

Cross-Covariance. Since $\mu_X = \mu_Y = 0$, $C_{XY}(\tau) = R_{XY}(\tau)$.

Normalized Cross-Covariance:

$$\rho_{XY}(\tau) = \frac{C_{XY}(\tau)}{\sigma^2} = \frac{-\sigma^2 \sin[\omega\tau]}{\sigma^2} = -\sin[\omega\tau]$$

Example 19.2.10 A random telegraph wave $X(t)$ as in Example 19.2.7 is passed through a linear system with impulse response $h(t) = e^{-\beta t}u(t)$, where $u(t)$ is a unit step function. The output of the system is $Y(t)$. It is desired to find the cross-correlation function $R_{XY}(\tau)$.

From Example 19.2.7 the autocorrelation function $R_X(\tau) = \frac{1}{4}\{1 + e^{-2\lambda|\tau|}\}$. Hence, from Eq. (19.2.23), we have

$$R_{XY}(\tau) = \int_0^\infty \frac{1}{4}(1 + e^{-2\lambda|\tau-\alpha|})e^{-\beta\alpha} d\alpha$$

$$= \begin{cases} \dfrac{1}{4\beta} + \dfrac{1}{4}\displaystyle\int_0^\tau e^{-2\lambda(\tau-\alpha)}e^{-\beta\alpha} d\alpha + \dfrac{1}{4}\displaystyle\int_\tau^\infty (e^{-2\lambda(\alpha-\tau)})e^{-\beta\alpha} d\alpha & \tau > 0 \\ \dfrac{1}{4\beta} + \dfrac{1}{4}\displaystyle\int_0^\infty e^{-2\lambda(\alpha-\tau)}e^{-\beta\alpha} d\alpha & \tau \le 0 \end{cases}$$

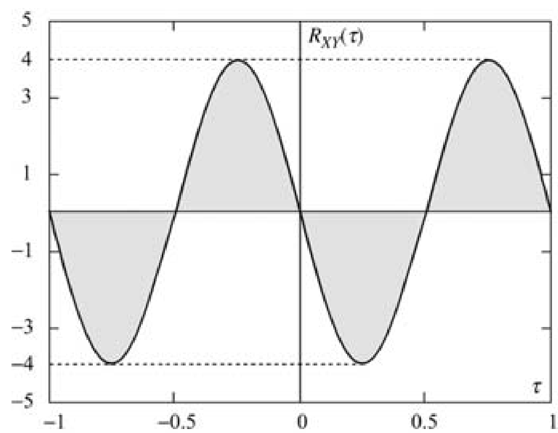

FIGURE 19.2.8

Evaluating the integrals and substituting $\lambda = 1$ and $\beta = 0.5$, we obtain

$$R_{XY}(\tau) = \frac{1}{4\beta} + \begin{cases} \dfrac{\lambda e^{-\beta\tau}}{4\lambda^2 - \beta^2} - \dfrac{e^{-2\lambda\tau}}{4(2\lambda - \beta)}, & \tau > 0 \\[2ex] \dfrac{e^{2\lambda\tau}}{4(2\lambda + \beta)}, & \tau \le 0 \end{cases}$$

or

$$R_{XY}(\tau) = \frac{1}{2} + \begin{cases} \dfrac{4}{15} e^{-(\tau/2)} - \dfrac{1}{6} e^{-2\tau}, & \tau > 0 \\[2ex] \dfrac{1}{10} e^{2\tau}, & \tau \le 0 \end{cases}$$

The cross-correlation function $R_{XY}(\tau)$ is shown in Fig. 19.2.9.
$R_{XY}(\tau)$ does not possess any symmetry, unlike $R_X(\tau)$.

Moments of Discrete-Time Stationary Processes

In actual practice observations are made on the sample function of a stationary random process $X(t)$ at equally spaced time intervals $\{t_i, i = 0, \pm 1, \ldots\}$ with corresponding sequence of random variables $\{X_i, i = 0, \pm 1, \ldots\}$. These observation random variables will *not be independent*. Since $\{X_i\}$ are samples of a stationary random process, the means and variances of these samples are the same as in the original process:

$$E[X_i] = \mu_X; \quad \text{var}[X_i] = \sigma_X^2, \quad i = 0, \pm 1, \ldots \tag{19.2.32}$$

Analogous to the continuous case, we can define the various second moments:

Autocovariance:

$$C_X(h) = E[(X_i - \mu_X)(X_{i+h} - \mu_X)], \quad i = 0, \pm 1, \ldots \tag{19.2.33}$$

Autocorrelation:

$$R_X(h) = E[X_i X_{i+h}], \quad i = 0, \pm 1, \ldots \tag{19.2.34}$$

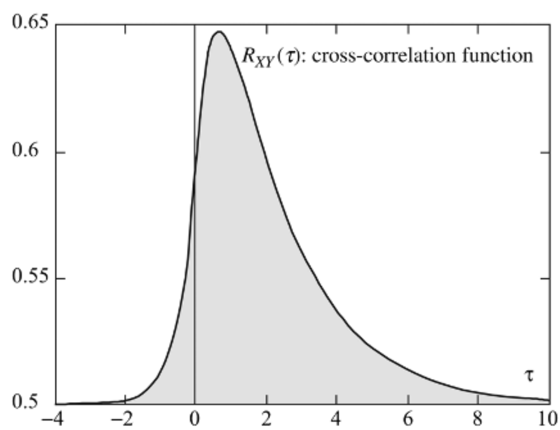

FIGURE 19.2.9

Normalized Autocovariance (*NACF*):

$$\rho_X(h) = \frac{C_X(h)}{C_X(0)} = \frac{C_X(h)}{\sigma_X^2} \quad (19.2.35)$$

If $\{Y_i,\ i = 0,\ \pm 1, \ldots\}$ is the sequence obtained from a second stationary random process $Y(t)$ with mean μ_Y and variance σ_Y^2, then we can define the cross-moments as follows:

Cross-Covariance:

$$C_{XY}(h) = E[(X_i - \mu_X)(Y_{i+h} - \mu_Y] \quad i = 0, \pm 1, \ldots \quad (19.2.36)$$

Cross-Correlation:

$$R_{XY}(h) = E[X_i Y_{i+h}] \quad i = 0, \pm 1, \ldots \quad (19.2.37)$$

Normalized Cross-Covariance (*NCCF*):

$$\rho_{XY}(h) = \frac{C_{XY}(h)}{\sqrt{C_X(0)C_Y(0)}} = \frac{C_{XY}(h)}{\sigma_X \sigma_Y} \quad (19.2.38)$$

Example 19.2.11 A discrete zero mean stationary random process X_i is given by

$$X_i = \phi X_{i-1} + v_i, \quad i = 1, \ldots \quad (19.2.39)$$

where v_i is a zero mean Gaussian random process with variance σ_v^2. We want to find the variance of X_i and the NACF $\rho_X(h)$.

Multiplying both sides of Eq. (19.2.39) by X_i and taking expectations, we have

$$E[X_i^2] = \phi E[X_i X_{i-1}] + E[X_i v_i] \quad \text{or} \quad \sigma_X^2 = \phi C_X(1) + E[X_i v_i] \quad (19.2.40)$$

The cross-correlation $E[X_i v_i]$ can be computed as follows:

$$E[X_i v_i] = E[(\phi X_{i-1} + v_i)v_i] = \sigma_v^2 \quad (19.2.41)$$

since v_i occurs after X_{i-1}. Hence, substituting Eq. (19.2.41) in Eq. (19.2.40) and dividing throughout by σ_X^2, we obtain

$$1 = \phi\rho_X(1) + \frac{\sigma_v^2}{\sigma_X^2} \quad \text{and} \quad \sigma_X^2 = \frac{\sigma_v^2}{1 - \phi\rho_X(1)} \quad (19.2.42)$$

Premultiplying both sides of Eq. (19.2.39) by X_{i-h} and taking expectations, we have

$$E[X_{i-h}X_i] = E[\phi X_{i-h}X_{i-1}] + E[X_{i-h}v_i] \quad (19.2.43)$$

In Eq. (19.2.43) $E[X_{i-h}v_i] = 0$ since v_i occurs after X_{i-h} for $h > 0$; hence

$$C_X(h) = \phi C_X(h-1), \quad h > 0 \quad (19.2.44)$$

Dividing Eq. (19.2.44) by $C_X(0) = \sigma_X^2$, we obtain an equation for the NACF $\rho_X(h)$

$$\rho_X(h) = \phi\rho_X(h-1), \quad h > 0 \quad (19.2.45)$$

and solving for $\rho_X(h)$ with initial condition $\rho_X(0) = 1$, we obtain

$$\rho_X(h) = \phi^h, \quad h > 0 \tag{19.2.46}$$

Substituting Eq. (19.2.46) in Eq. (19.2.42), we have

$$\sigma_X^2 = \frac{\sigma_v^2}{1 - \phi^2} \tag{19.2.47}$$

The process defined by Eq. (19.2.39) is called *an autoregressive process* of order 1.

19.3 ERGODIC PROCESSES

The ensemble average of a random process $X(t)$ is the mean value $\mu_X(t)$ defined by,

$$\mu_X(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx \tag{19.3.1}$$

Finding the ensemble average $\mu_X(t)$ requires storing a multiplicity of sample functions and finding the average. In many instances this process may be nontrivial. Given a sample function of any random process $X(t)$, the *time average* $\hat{\mu}_X$ is defined by

$$\hat{\mu}_X = \frac{1}{2T} \int_{-T}^{T} X(t) dt \tag{19.3.2}$$

A reasonable question to ask is whether the ensemble average can be obtained from a much easier time average. We observe that the ensemble average is not a random variable but is a function of time, whereas the time average is a random variable that is not a function of time. If these averages are to be equal, then the first condition that we have to impose is that the random process $X(t)$ be stationary, which removes the time factor in the mean value. Under these conditions, we can write

$$E\left[\frac{1}{2T} \int_{-T}^{T} X(t) dt\right] = E[\hat{\mu}_X] = \frac{1}{2T} \int_{-T}^{T} E[X(t)] dt = \mu_X \tag{19.3.3}$$

and $\hat{\mu}_X$ is an unbiased estimator of μ_X.

If a random variable is to be equal to a constant, then the second condition from Example 14.1.1 is that the variance of $\hat{\mu}_X$ must tend to zero as $T \to \infty$. Such random processes are said to satisfy the *ergodic hypothesis*. We will now derive the conditions for a random process to be *mean-ergodic* and *correlation-ergodic*.

Mean-Ergodic

A stationary random process $X(t)$ is called *mean-ergodic* if the ensemble average is equal to the time average of the sample function $x(t)$. We will assume that the following conditions are satisfied by $X(t)$:

$X(t)$ is stationary, implying that it has a constant mean μ_X, and the autocorrelation function $R_X(t, t + \tau)$ is a function of τ only, or $R_X(t, t + \tau) = R_X(\tau)$.

$R_X(0) = E[X^2(t)]$ is bounded, or $R_X(0) < \infty$. Hence $C_X(0) = R_X(0) - \mu_X^2 < \infty$.