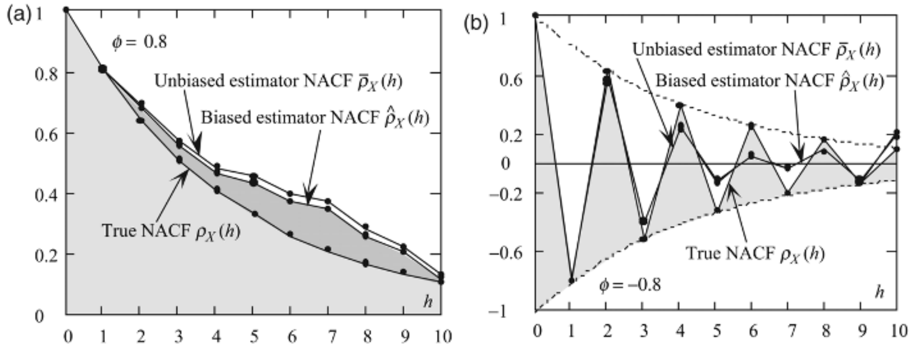

FIGURE 19.4.15

estimated NACFs $\bar{\rho}_X(h)$ and $\hat{\rho}_X(h)$ are determined from Eqs. (19.4.78) and (19.4.88) respectively. They are shown in Fig. 19.4.15 for $\phi = 0.8$ and -0.8 for 10 lags.

Here also the NACFs for $\phi = -0.8$ show oscillatory behavior. For the 10 lags shown,there is very little difference between the two estimators $\bar{\rho}_X(h)$ and $\hat{\rho}_X(h)$. However, the discrepancy between the true NACF and the estimators for higher lags is to be expected.

19.5 POWER SPECTRAL DENSITY

19.5.1 Continuous Time

In signal analysis power spectra are associated with Fourier transforms that transform signals from the time domain to the frequency domain. The same concept is also applicable to stationary random processes. The correlation functions represent stationary processes in the time domain. We can transform them to the frequency domain by taking their Fourier transforms. The *power spectral density* (psd) function $S_X(\omega)$ of a real stationary random process $X(t)$ is defined as the Fourier transform of the autocorrelation function:

$$S_X(\omega) = \mathrm{FT}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau \qquad (19.5.1)$$

From the Fourier inversion theorem we can obtain the autocorrelation function from the power spectral density:

$$R_X(\tau) = \mathrm{IFT}[S_X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega \qquad (19.5.2)$$

Equations (19.5.1) and (19.5.2) are called the *Wiener–Khinchine* theorem.

Since $R_X(0) = E[X^2(t)]$, the average power in the random process, we obtain the following from Eq. (19.5.2):

$$R_X(0) = E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \int_{-\infty}^{\infty} S_X(f)df \qquad (19.5.3)$$

Thus, $S_X(f)$ represents the average power per hertz, and hence the term *power spectral density*. Since $R_X(\tau)$ is an even function, we can rewrite Eq. (19.5.1) as

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)[\cos(\omega\tau) + j\sin(\omega\tau)]d\tau = \int_{-\infty}^{\infty} R_X(\tau)\cos(\omega\tau)d\tau \qquad (19.5.4)$$

and $S_X(\omega)$ is also an even function. Hence Eq. (19.5.2) can be rewritten as

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\cos(\omega\tau)d\omega \qquad (19.5.5)$$

The *cross-spectral density* (csd) $S_{XY}(\omega)$ of two real stationary random processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of the cross-correlation function, $R_{XY}(\tau)$

$$S_{XY}(\omega) = \text{FT}[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau \qquad (19.5.6)$$

and the inverse Fourier transform of $S_{XY}(\omega)$ gives the cross-correlation function:

$$R_{XY}(\tau) = \text{IFT}[S_{XY}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega \qquad (19.5.7)$$

The cross-spectral density $S_{XY}(\omega)$ will be complex, in general, even when the random processes $X(t)$ and $Y(t)$ are real.

The Fourier transforms can also be obtained from the tables in Appendix A.

Example 19.5.1 The power spectral density (psd) of the random binary wave of Example 19.2.6 is to be determined. The autocorrelation of the random process $X(t)$ is given by

$$R_X(\tau) = \begin{cases} A^2\left(1 - \dfrac{|\tau|}{T}\right), & |\tau| < T \\ 0, & \text{otherwise} \end{cases}$$

The psd $S_X(\omega)$ given by

$$\begin{aligned} S_X(\omega) &= \int_{-T}^{0} A^2\left(1 + \frac{\tau}{T}\right)e^{-j\omega\tau}d\tau + \int_{0}^{T} A^2\left(1 - \frac{\tau}{T}\right)e^{-j\omega\tau}d\tau \\ &= \frac{A^2}{\omega^2 T}[(1 - e^{j\omega T} + j\omega T) + (1 - e^{-j\omega T} - j\omega T)] \\ &= A^2 T\left[\frac{\sin(\omega T/2)}{\omega T/2}\right]^2 \end{aligned}$$

is shown in Fig. 19.5.1. As we can see from figure, the psd is an even function.

Example 19.5.2 We will find the psd from the autocorrelation $R_X(\tau) = \frac{1}{4}[1 + e^{-2\lambda|\tau|}]$ of the random telegraph wave given in Example 19.2.7. Taking the Fourier transform, the psd

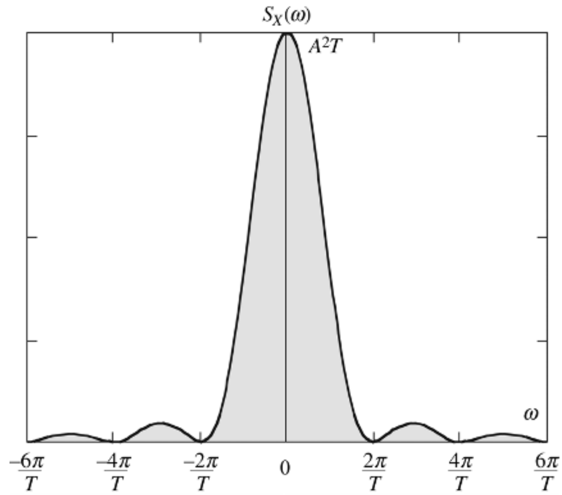

FIGURE 19.5.1

is given by

$$S_X(\omega) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-j\omega\tau}d\tau + \frac{1}{4}\int_{-\infty}^{0} e^{2\lambda\tau}e^{-j\omega\tau}d\tau + \frac{1}{4}\int_{0}^{\infty} e^{-2\lambda\tau}e^{-j\omega\tau}d\tau$$

$$= \frac{1}{4}\left[2\pi\delta(\omega) + \frac{4\lambda}{4\lambda^2 + \omega^2}\right] = \frac{\pi}{2}\delta(\omega) + \frac{\lambda}{4\lambda^2 + \omega^2}$$

The impulse function $(\pi/2)\delta(\omega)$ represents the direct-current (dc) value of $R_X(\tau) = \frac{1}{4}$. The psd is shown in Fig. 19.5.2 for $\lambda = 1$.

Example 19.5.3 We will find the psd from the autocorrelation $R_X(\tau) = e^{-2\lambda|\tau|}\cos(\omega_0\tau)$ of Example 19.2.8. We can take the Fourier transform of $R_X(\tau)$ directly, but it is easier to

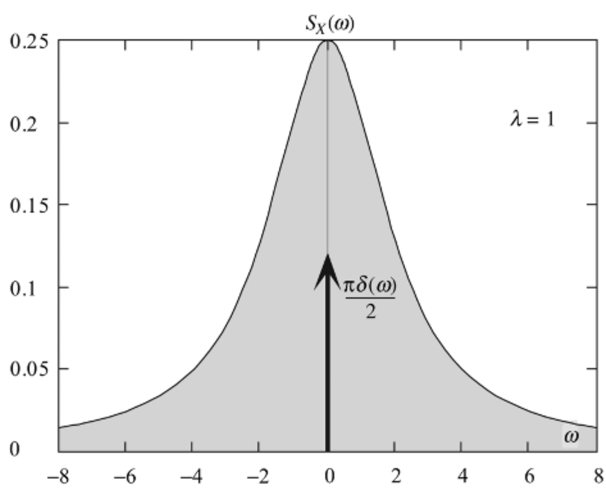

FIGURE 19.5.2

evaluate the FT by using the frequency convolution property of the FT as follows:

$$\text{FT}[e^{-2\lambda|\tau|} = \frac{4\lambda^2}{4\lambda^2 + \omega^2} \quad \text{and} \quad \text{FT}[\cos(\omega_0\tau)] = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

Using the frequency convolution property $x(t)y(t) \Leftrightarrow (1/2\pi)X(\omega) * Y(\omega)$, we have

$$e^{-2\lambda|\tau|}\cos(\omega_0\tau) \iff \frac{1}{2\pi}\frac{4\lambda^2}{4\lambda^2 + \omega^2} * \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

or

$$S_X(\omega) = 2\lambda\left[\frac{1}{4\lambda^2 + (\omega + \omega_0)^2} + \frac{1}{4\lambda^2 + (\omega - \omega_0)^2}\right]$$

and simplifying, we obtain

$$S_X(\omega) = \frac{4\lambda(\omega^2 + \omega_0^2 + 4\lambda^2)}{\omega^4 - 2(\omega_0^2 - 4\lambda^2)\omega^2 + (\omega_0^2 + 4\lambda^2)^2}$$

With $\lambda = 1$ and $\omega_0 = 2\pi$, the psd becomes

$$S_X(\omega) = \frac{4(\omega^2 + 4\pi^2 + 4\lambda^2)}{\omega^4 - 8(\pi^2 - 1)\omega^2 + 16(\pi^2 + 1)^2}$$

The psd $S_X(\omega)$ is shown in Fig. 19.5.3.

Example 19.5.4 (Bandlimited Process) The autocorrelation function $R_X(\tau) = \sin(\omega_0\tau)/\pi\tau$ of a stationary random process $X(t)$ is shown in Fig. 19.5.4 for $\omega_0 = 2\pi$. We have to find the psd $S_X(\omega)$.

From item 2 in the FT table, the Fourier transform of $R_X(\tau)$ can be obtained as follows:

$$\frac{\sin(\omega_0\tau)}{\pi\tau} \iff p_{\omega_0}(\omega), \quad \text{hence} \quad S_X(\omega) = p_{\omega_0}(\omega)$$

The psd $S_X(\omega)$ is shown in Fig. 19.5.5 for $\omega_0 = 2\pi$.

The random process that has a psd as in Fig. 19.5.5 is called a *bandlimited signal* since the frequency spectrum exists only between -2π and 2π.

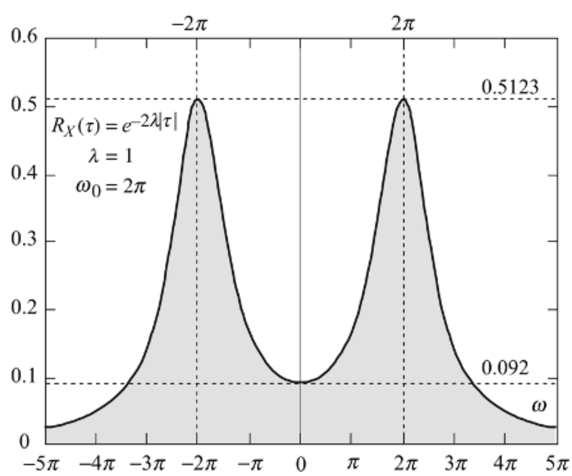

FIGURE 19.5.3

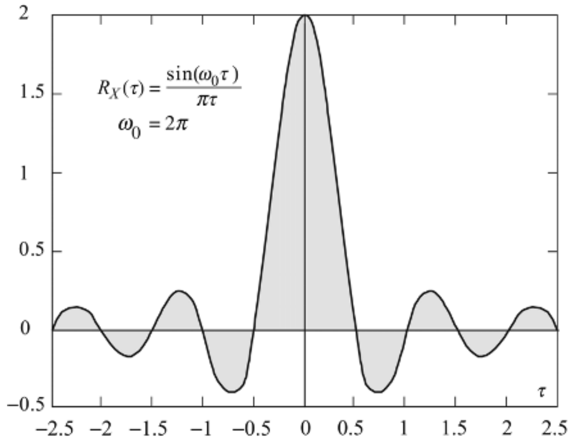

FIGURE 19.5.4

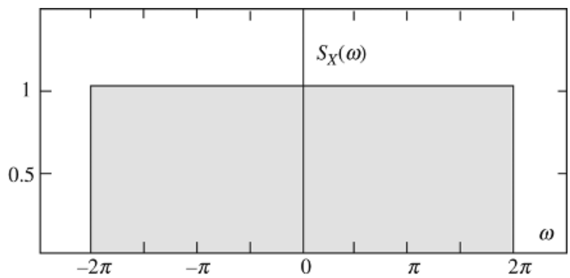

FIGURE 19.5.5

Example 19.5.5 (Bandlimited Process) The autocorrelation function $R_X(\tau)$

$$R_X(\tau) = \frac{\sin[\omega_0(\tau - \tau_0)]}{\pi(\tau - \tau_0)} + \frac{\sin[\omega_0(\tau + \tau_0)]}{\pi(\tau + \tau_0)}$$

of a bandlimited random process $X(t)$ is shown in Fig. 19.5.6 for $\omega_0 = 2\pi$ and $\tau_0 = 3$.

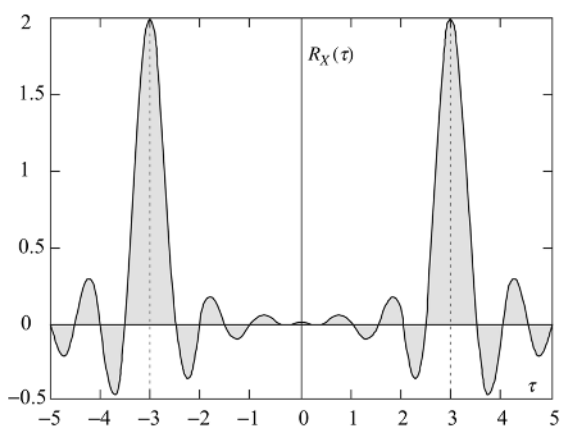

FIGURE 19.5.6

Since $\sin(\omega_0\tau)/\pi\tau \Leftrightarrow p_{\omega_0}(\omega)$ using the time-shifting property $x(t \pm t_0) \Leftrightarrow X(\omega)e^{\pm j\omega t_0}$ of the FT, we have

$$\frac{\sin[\omega_0(\tau-\tau_0)]}{\pi(\tau-\tau_0)} \iff p_{\omega_0}(\omega)e^{-j\omega\tau_0}, \quad \frac{\sin[\omega_0(\tau+\tau_0)]}{\pi(\tau+\tau_0)} \iff p_{\omega_0}(\omega)e^{j\omega\tau_0}$$

or

$$\frac{\sin[\omega_0(\tau-\tau_0)]}{\pi(\tau-\tau_0)} + \frac{\sin[\omega_0(\tau+\tau_0)]}{\pi(\tau+\tau_0)} \iff p_{\omega_0}(\omega)[e^{j\omega\tau_0} + e^{-j\omega\tau_0}]$$

and

$$S_X(\omega) = 2p_{\omega_0}(\omega)\cos(\omega\tau_0)$$

The psd is shown in Fig. 19.5.7 for $\omega_0 = 2\pi$ and $\tau_0 = 3$.

Example 19.5.6 (Bandpass Process) The autocorrelation function $R_X(\tau)$

$$R_X(\tau) = \frac{2\sin(\omega_0\tau)}{\pi\tau}\cos(\omega_c\tau)$$

of a bandpass random process $X(t)$ is shown in Fig. 19.5.8 for $\omega_0 = 2\pi$ and $\omega_c = 16\pi$. We will find the psd using the frequency convolution property of the FT:

$$x(t)y(t) \iff \frac{1}{2\pi}X(\omega) * Y(\omega)$$

Since $2\sin(\omega_0\tau)/\pi\tau \Leftrightarrow 2p_{\omega_0}(\omega)$ and $\cos(\omega_c\tau) \Leftrightarrow \pi[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)]$, we have

$$\frac{2\sin(\omega_0\tau)}{\pi\tau}\cos(\omega_c\tau) \iff \frac{1}{2\pi}2p_{\omega_0}(\omega) * \pi[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)]$$

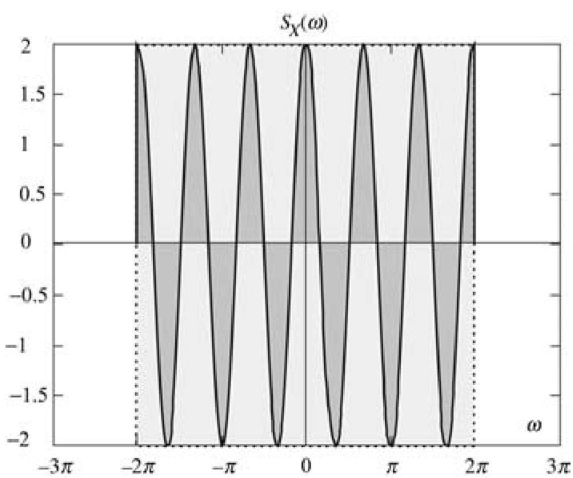

FIGURE 19.5.7

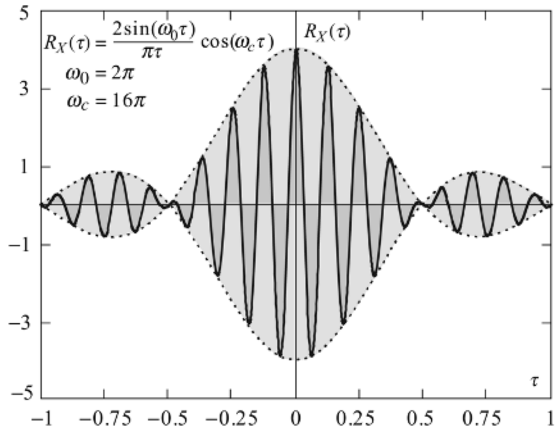

FIGURE 19.5.8

and the psd

$$S_X(\omega) = p_{\omega_0}(\omega + \omega_c) + p_{\omega_0}(\omega - \omega_c)$$

The function $S_X(\omega)$ is shown in Fig. 19.5.9 for $\omega_0 = 2\pi$ and $\omega_c = 16\pi$.

Example 19.5.7 (White Noise) A white-noise process has the autocorrelation function $R_X(\tau) = \sigma_X^2\delta(\tau)$ given in Eq. (19.2.17). The psd of white noise is given by

$$S_X(\omega) = \sigma_X^2, \quad -\infty < \omega < \infty$$

and has a flat spectrum. Since the process has all frequencies with equal power, it is called "white noise," analogously to white light. It has infinite energy since $R_X(0) = (1/2\pi)\int_{-\infty}^{\infty}\sigma_X^2 d\omega = \infty$, and hence it is an idealization.

Example 19.5.8 The cross-correlation function of two random processes $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ and $Y(t) = -A\sin(\omega_0 t) + B\cos(\omega_0 t)$ in Example 19.2.9 is $R_{XY}(\tau) = -\sigma^2\sin(\omega_0\tau)$, $E[A^2] = E[B^2] = \sigma^2$. We will find the cross-spectral density $S_{XY}(\omega)$.

Taking FT of $-\sigma^2\sin(\omega_0\tau)$, from tables, we have

$$S_{XY}(\omega) = -j\pi\sigma^2[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

and this is shown in Fig. 19.5.10.

Note that $S_{XY}(\omega)$ is neither even nor real.

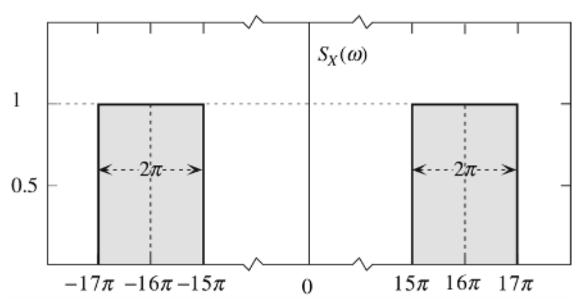

FIGURE 19.5.9

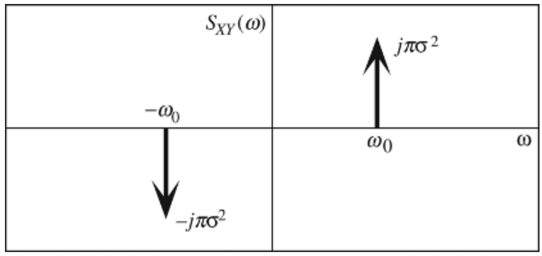

FIGURE 19.5.10

Example 19.5.9 The cross-spectral density is to be found for the cross-correlation function $R_{XY}(\tau)$ derived in Example 19.2.10:

$$R_{XY}(\tau) = \frac{1}{2} + \begin{cases} \frac{4}{15} e^{-(\tau/2)} - \frac{1}{6} e^{-2\tau}, & \tau > 0 \\ \frac{1}{10} e^{2\tau}, & \tau \leq 0 \end{cases}$$

Taking FT of each term in $R_{XY}(\tau)$, we have

$$\frac{4}{15} e^{(\tau/2)} \Longleftrightarrow \frac{4}{15} \frac{1}{1/2 + j\omega}; \quad \frac{1}{6} e^{-2\tau} \Longleftrightarrow \frac{1}{6} \frac{1}{2 + j\omega}$$

$$\frac{1}{2} \Longleftrightarrow \pi\delta(\omega); \quad \frac{1}{10} e^{2\tau} \Longleftrightarrow \frac{1}{10} \frac{1}{2 - j\omega}$$

The cross-spectral density $S_{XY}(\omega)$ is obtained by adding the FT terms shown above:

$$S_{XY}(\omega) = \pi\delta(\omega) + \frac{4}{15} \frac{1}{\frac{1}{2} + j\omega} + \frac{1}{6} \frac{1}{2 + j\omega} + \frac{1}{10} \frac{1}{2 - j\omega} = \frac{1}{3} \frac{8 + 2\omega^2 + 3j\omega}{(1 + 2j\omega)(4 + \omega^2)} + \pi\delta(\omega)$$

The cross-spectral density $S_{XY}(\omega)$ is complex and possesses no symmetry, unlike the power spectral density. The constant term $\frac{1}{2}$ in $R_{XY}(\tau)$ gives rise to the impulse function in the frequency domain. The amplitude $|S_{XY}(\omega)|$ and the phase $\arg\{S_{XY}(\omega)\}$ spectra are graphed in Fig. 19.5.11.

Properties of Power Spectral Densities of Stationary Random Processes

1. $S_X(\omega)$ is a real function. In general, the Fourier transform $X(\omega)$ of any function $X(t)$ will be complex. However, the ACF $R_X(\tau)$ is an even function and satisfies the relation $R_X(\tau) = R_X(-\tau)$. Hence, from the definition of psd, we obtain

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_X(\tau)[\cos(\omega\tau) + j\sin(\omega\tau)] d\tau$$
$$= \int_{-\infty}^{\infty} R_X(\tau)\cos(\omega\tau) d\tau$$

since the imaginary part $\int_{-\infty}^{\infty} R_X(\tau) j\sin(\omega\tau) d\tau = 0$ because an even function multiplying an odd function results in an odd function and the integral of an odd function over $(-\infty,\infty)$ is zero.

2. From property 1 the psd $S_X(\omega)$ is an even function and, hence it is a function of ω^2. As a consequence $S_X(\omega) = S_X(-\omega)$. (19.5.8)

3. $S_X(\omega) \geq 0$: The psd is a nonnegative function of ω. (19.5.9)

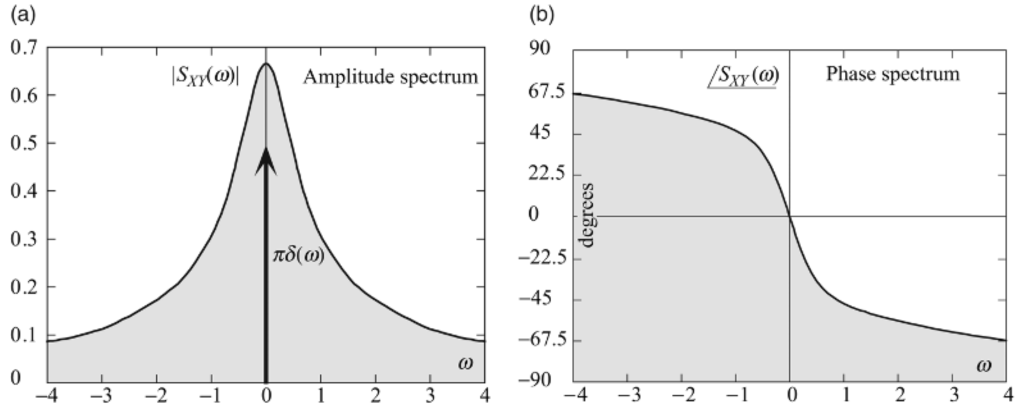

FIGURE 19.5.11

4. From the Fourier transform properties:

$$R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega = \int_{-\infty}^{\infty} S_X(f)df = E[X^2(t)]$$
$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau \tag{19.5.10}$$

5. If $Z(t) = X(t) + Y(t)$, then from Eq. (19.2.30), we have

$$\begin{aligned} R_Z(\tau) &= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau) \\ S_Z(\omega) &= S_X(\omega) + S_Y(\omega) + S_{XY}(\omega) + S_{YX}(\omega) \end{aligned} \tag{19.5.11}$$

If $X(t)$ and $Y(t)$ are orthogonal, then Eq. (19.5.11) reduces to $S_Z(\omega) = S_X(\omega) + S_Y(\omega)$

Alternate Form for Power Spectral Density

The psd can also be obtained directly from the stationary random process $X(t)$. We will truncate $X(t)$ in the interval $(-T, T)$ and define the random process $X_T(t)$ as

$$X_T(t) = \begin{cases} X(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{19.5.12}$$

The FT $X_T(\omega)$ of $X_T(t)$, given by $X_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}dt$, is a random variable. The quantity $S_T(\omega)$, defined by

$$S_T(\omega) = \frac{1}{2T}\left|\int_{-T}^{T} X(t)e^{-j\omega t}dt\right|^2 = \frac{|X_T(\omega)|^2}{2T} = \frac{X_T(\omega)X_T^*(\omega)}{2T} \tag{19.5.13}$$

is called the *periodogram*. The periodogram represents the power of the sample function $X(t)$ at the frequency ω. Equation (19.5.13) can be expanded as follows:

$$S_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} X(t)X(s)e^{-j\omega(t-s)}dt\,ds \tag{19.5.14}$$

We can make the transformation $\tau = t - s$ and $u = s$ in Eq. (19.5.14) and perform the integration in the (τ, u) plane. The Jacobian $\| J \|$ of the transformation from Eq. (19.3.5) is 1, and Eq. (19.5.14) can be transformed as follows:

$$S_T(\omega) = \frac{1}{2T}\int\left[\int X(u)X(u+|\tau|)du\right]e^{-j\omega\tau}d\tau \tag{19.5.15}$$

We can obtain the limits of integration from Fig. 19.3.1 and write Eq. (19.5.15) as

$$\begin{aligned} S_T(\omega) &= \int_0^{2T}\left[\frac{1}{2T}\int_{-T+\tau}^{T} X(u)X(u+\tau)du\right]e^{-j\omega\tau}d\tau \\ &\quad + \int_{-2T}^{0}\left[\frac{1}{2T}\int_{-T}^{T-|\tau|} X(u)X(u+\tau)du\right]e^{-j\omega\tau}d\tau \\ &= \int_0^{2T}\hat{R}_X(\tau)e^{-j\omega\tau}d\tau + \int_{-2T}^{0}\hat{R}_X(\tau)e^{-j\omega\tau}d\tau \\ &= \int_{-2T}^{2T}\hat{R}_X(\tau)e^{-j\omega\tau}d\tau \end{aligned} \tag{19.5.16}$$

where we have used the definition for $\hat{R}_X(\tau)$ as in Eq. (19.4.34a). Using Eq. (19.4.34b), the expected value of $S_T(\omega)$ can be written as

$$E[S_T(\omega)] = \int_{-2T}^{2T} E\left[\hat{R}_X(\tau)\right]e^{-j\omega\tau}d\tau = \int_{-2T}^{2T} R_X(\tau)\left(1-\frac{|\tau|}{2T}\right)e^{-j\omega\tau}d\tau \tag{19.5.17}$$

Taking the limit of $E[S_T(\omega)]$ in Eq. (19.5.17) as $T \to \infty$, we have

$$\begin{aligned} \lim_{T\to\infty} E[S_T(\omega)] &= \lim_{T\to\infty}\left\{\int_{-\infty}^{\infty} R_X(\tau)\left(1-\frac{|\tau|}{2T}\right)e^{-j\omega\tau}d\tau\right\} \\ &= \lim_{T\to\infty} \mathrm{FT}\left[R_X(\tau)\left(1-\frac{|\tau|}{2T}\right)\right] \end{aligned} \tag{19.5.18}$$

Using the frequency convolution property of the FT, we obtain

$$x(t)y(t) \iff \frac{1}{2\pi}X(\omega) * Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(p)Y(\omega-p)dp$$

and with $R_X(\tau) \Leftrightarrow S_X(\omega)$ and

$$\left(1-\frac{|\tau|}{2T}\right) \Leftrightarrow 2T\frac{\sin^2 \omega T}{(\omega T)^2}$$

Eq. (19.5.18) can be rewritten as follows:

$$\begin{aligned} \lim_{T\to\infty} E[S_T(\omega)] &= \lim_{T\to\infty} \mathrm{FT}\left[R_X(\tau)\left(1-\frac{|\tau|}{2T}\right)\right] \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty} S_X(p)T\frac{\sin^2(\omega-p)T}{\pi[(\omega-p)T]^2}dp \end{aligned} \tag{19.5.19}$$

From Ref. 36,

$$\lim_{T\to\infty} T\frac{\sin^2(\omega T)}{\pi(\omega T)^2} \longrightarrow \delta(\omega) \quad \text{and} \quad \lim_{T\to\infty} T\frac{\sin^2(\omega - p)T}{\pi[(\omega - p)T]^2} = \delta(\omega - p)$$

and substituting this result in Eq. (19.5.19), we have

$$\lim_{T\to\infty} E[S_T(\omega)] = \int_{-\infty}^{\infty} S_X(p)\delta(\omega - p)dp = S_X(\omega) \tag{19.5.20}$$

Thus, the expected value of the periodogram as $T \to \infty$ yields the psd $S_X(\omega)$.

Estimation of Power Spectral Density

We have defined an estimator $\hat{R}_X(\tau)$ for the autocorrelation function in Eq. (19.4.34a). It is intuitive to express the estimator $\hat{S}_X(\omega)$ for the psd as the FT$[\hat{R}_X(\tau)]$ as follows:

$$\hat{S}_{XT}(\omega) = \int_{-T}^{T} \hat{R}_X(\tau)e^{-j\omega\tau}\mathrm{d}\tau : \hat{S}_X(\omega) = \lim_{T\to\infty} \hat{S}_{XT}(\omega) \tag{19.5.21}$$

This equation corresponds to Eq. (19.5.16). The periodogram is an asymptotically unbiased estimator for $S_X(\omega)$ since $\lim_{T\to\infty} E[\hat{S}_X(\omega)] = S_X(\omega)$. We may be tempted to conclude that since $\hat{R}_X(\tau)$ is a consistent estimator of $R_X(\tau)$, the Fourier transform of $\hat{R}_X(\tau)$ will also be a consistent estimator of $S_X(\omega)$. This is not true because $\hat{S}_X(\omega)$ fails to converge to $S_X(\omega)$ for $T \to \infty$. We will show heuristically that $\mathrm{var}[\hat{S}_{X_T}(\omega)]$ does not approach 0 as $T \to \infty$. From Eq. (19.5.14) the second moment $E[\hat{S}_{X_T}(\omega)]^2$ is given by

$$E\left[\hat{S}_{X_T}(\omega)\right]^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T} E[X(t)X(s)X(v)X(u)] \times e^{-j\omega(t-s+v-u)}dt\,ds\,dv\,du \tag{19.5.22}$$

If the process X(t) is Gaussian, then from Eq. (11.6.8) or Example 19.3.4, we obtain

$$E[X(t)X(s)X(v)X(u)] = R_X(t-s)R_X(v-u) + R_X(t-v)R_X(s-u) + R_X(t-u)R_X(s-v) \tag{19.5.23}$$

Substituting Eq. (19.5.23) into Eq. (19.5.22), we have

$$E[\hat{S}_{X_T}(\omega)]^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T}[R_X(t-s)R_X(v-u) + R_X(t-u)R_X(s-u) + R_X(t-u)R_X(s-v)] \times e^{-j\omega(t-s+v-u)}dt\,ds\,dv\,du \tag{19.5.24}$$

Rearranging Eq. (19.5.24), we obtain

$$\begin{aligned}E[\hat{S}_{X_T}(\omega)]^2 = &\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(t-s)e^{-j\omega(t-s)}dt\,ds\\ &\times\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(v-u)e^{-j\omega(v-u)}dv\,du\\ &+\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(t-u)e^{-j\omega(t-u)}dt\,du\\ &\times\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(v-s)e^{-j\omega(v-s)}dv\,ds\\ &+\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(t-v)e^{-j\omega(t+v)}dt\,dv\\ &\times\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(s-u)e^{j\omega(s+u)}ds\,du \end{aligned}\tag{19.5.25}$$

Using Eq. (19.5.21), we can simplify Eq. (19.5.25) as follows:

$$E[\hat{S}_{X_T}(\omega)]^2 = 2\{E[\hat{S}_{X_T}(\omega)]\}^2 + \left\{\frac{1}{4T^2}\left|\int_{-T}^{T}\int_{-T}^{T} R_X(t-v)e^{-j\omega(t+v)}dt\,dv\right|^2\right\}\tag{19.5.26}$$

As $T \to \infty$, the second term on the righthand side of Eq. (19.5.26) tends to zero. Hence

$$E[\hat{S}_{X_T}(\omega)]^2 \geq 2\{E[\hat{S}_{X_T}(\omega)]\}^2, \quad \omega \neq 0$$

Subtracting $\{E[\hat{S}_{X_T}(\omega)]\}^2$ from both sides of this equation, we obtain

$$E[\hat{S}_{X_T}(\omega)]^2 - \{E[\hat{S}_{X_T}(\omega)]\}^2 \geq \{E[\hat{S}_{X_T}(\omega)]\}^2, \quad \omega \neq 0$$

or

$$\text{var}[\hat{S}_{X_T}(\omega)] \geq \{E[\hat{S}_{X_T}(\omega)]\}^2, \quad \omega \neq 0 \tag{19.5.27}$$

Substitution of Eq. (19.5.20) $\lim_{T\to\infty} E[S_{X_T}(\omega)] = S_X(\omega)$, in Eq. (19.5.27) results in

$$\lim_{T\to\infty} \text{var}[\hat{S}_{X_T}(\omega)] \approx S_X^2(\omega), \quad \omega \neq 0 \tag{19.5.28}$$

Thus, as $T \to \infty$, $\text{var}[\hat{S}_{X_T}(\omega)]$ does not go to zero but is approximately equal to the psd $S_X^2(\omega)$, and hence $\hat{S}_X(\omega)$ is not a consistent estimator. To make $\hat{S}_X(\omega)$ consistent, spectral windowing [25] is employed. However, with spectral windowing the asymptotically unbiased nature of the estimator is lost.

19.5.2 Discrete Time

As in Eq. (19.5.1), we can also define power spectral density for a discrete-time stationary ergodic random process $\{X_i = X(t_i),\ i = 0,\ \pm 1, \ldots\}$, where the intervals are equally spaced. The autocorrelation function $R_X(h)$ has been defined in Eq. (19.2.34). The psd $S_X(\omega)$ is the Fourier transform of $R_X(h)$ and is given by

$$S_X(\omega) = \text{FT}[R_X(h)] = \sum_{h=-\infty}^{\infty} R_X(h)e^{-j\omega h} \tag{19.5.29}$$

Since the discrete-time Fourier transforms are periodic, the psd for discrete-time random process X_i as given by Eq. (19.5.29) is periodic with period 2π.

The inverse FT of $S_X(\omega)$ is the autocorrelation function $R_X(h)$, given by

$$R_X(h) = \text{IFT}[S_X(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\omega)e^{j\omega h}d\omega \tag{19.5.30}$$

Example 19.5.10 (Discrete Analog of Example 19.5.1) The autocorrelation $R_X(h)$ of the discrete-time process $\{X_i\}$ corresponding to the continuous-time process $X(t)$ of Example 19.5.1 is given by

$$R_X(h) = \begin{cases} A^2\left(1 - \dfrac{|h|}{n}\right), & |h| < n \\ 0, & \text{otherwise} \end{cases}$$

and is shown in Fig. 19.5.12 for $A = 1$ and $n = 10$.

The psd is obtained from Eq. (19.5.29) for finite n as

$$\begin{aligned} S_X(\omega) &= \sum_{h=-(n-1)}^{n-1} A^2\left(1 - \frac{|h|}{n}\right)e^{-j\omega h} \\ &= A^2\left[1 + \sum_{h=-(n-1)}^{-1}\left(1 + \frac{h}{n}\right)e^{-j\omega h} + \sum_{h=1}^{n-1}\left(1 - \frac{h}{n}\right)e^{-j\omega h}\right] \\ &= A^2\left[1 + \sum_{h=1}^{n-1}\left(1 - \frac{h}{n}\right)\left(e^{j\omega h} + e^{-j\omega h}\right)\right] = A^2\left[1 + 2\sum_{h=1}^{n-1}\left(1 - \frac{h}{n}\right)\cos(\omega h)\right] \\ &= \frac{A^2}{n}\left[\frac{1 - \cos(n\omega)}{1 - \cos(\omega)}\right] = \frac{A^2}{n}\left[\frac{\sin(n\omega/2)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

and

$$S_X(0) = \frac{A^2 n^2}{n} = A^2 n$$

The psd is graphed in Fig. 19.5.13 for $A = 1$ and $n = 5$ for values of ω between -2π and 2π, and the periodic nature of the psd is evident.

Figure 19.5.13 is similar to Fig. 19.5.1 except for the periodicity.

Example 19.5.11 (Discrete Analog of Example 19.5.2) The discrete AC function $R_X(h)$ corresponding to Example 19.5.2 is given by

$$R_X(h) = \sigma_X^2\left[1 + e^{-2\lambda|h|}\right], \quad -\infty < h < \infty$$

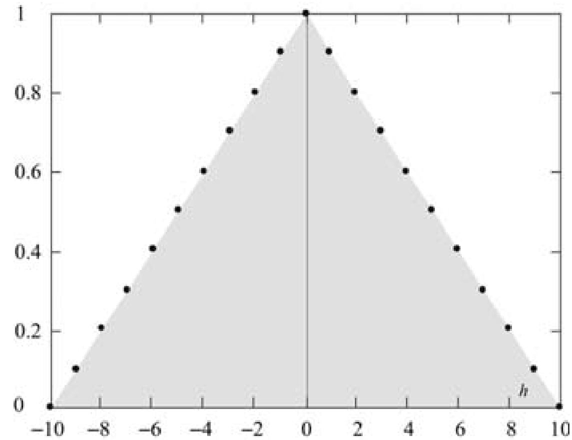

FIGURE 19.5.12

where 1 is a discrete unity function. The psd is determined as follows:

$$\begin{aligned} S_X(\omega) &= \sum_{h=-\infty}^{\infty} \sigma_X^2\big(1 + e^{-2\lambda|h|}\big)e^{-j\omega h} \\ &= \sigma_X^2\left[\sum_{h=-\infty}^{\infty} 1 \,.\, e^{-j\omega h} + \sum_{h=-\infty}^{-1} e^{2\lambda h}e^{-j\omega h} + \sum_{h=1}^{\infty} e^{-2\lambda h}e^{-j\omega h}\right] \\ &= \sigma_X^2\left\{\sum_{h=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi h) + 1 + \sum_{h=1}^{\infty}\left[e^{-h(2\lambda+j\omega)} + e^{-h(2\lambda-j\omega)}\right]\right\} \\ &= \sigma_X^2\left\{\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi h) + \frac{1 - e^{-4\lambda}}{1 + e^{-4\lambda} - 2e^{-2\lambda}\cos(\omega)}\right\} \end{aligned}$$

where $\delta(\omega)$ is the Dirac delta function. The delta function train corresponds the discrete constant $\sigma_X^2 = \frac{1}{4}$ in the frequency domain. The plot of $S_X(\omega)$ is shown in Fig. 19.5.14

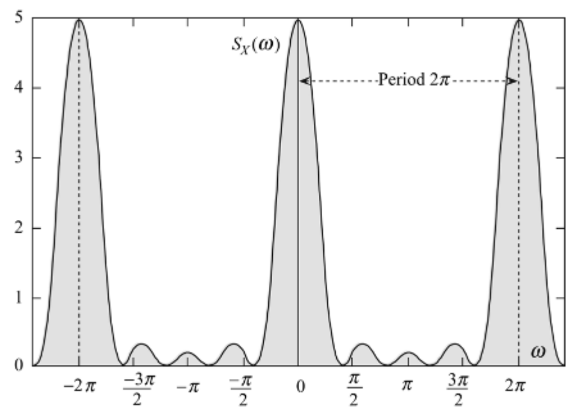

FIGURE 19.5.13

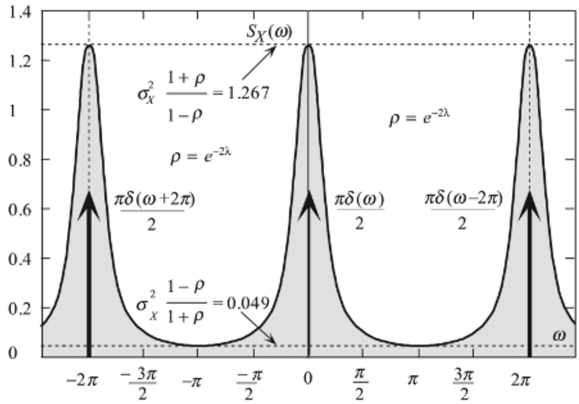

FIGURE 19.5.14

with $\sigma_X^2 = \frac{1}{5}$ and $\lambda = \frac{1}{5}$. The graph is very similar to Fig. 19.5.2 except that it is periodic with period equal to 2π.

Alternate Form for Power Spectral Density

The discrete psd can also be obtained from the stationary discrete-time random process $\{X_k\}$. We will truncate this process in the interval $(0, N-1)$ and write

$$X_{kN} = \begin{cases} X_k, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{19.5.31}$$

The discrete-time FT of the sequence $\{X_{kN}\}$ is given by $X_N(\omega) = \sum_{k=0}^{N-1} X_k e^{-j\omega k}$, which is a random variable. We now define the quantity $S_N(\omega)$ as

$$S_N(\omega) = \frac{1}{N}\left|\sum_{k=0}^{N-1} X_k e^{-j\omega k}\right|^2 = \frac{|X_N(\omega)|^2}{N} = \frac{X_N(\omega)X_N^*(\omega)}{N} \tag{19.5.32}$$

which is the periodogram for the discrete random process. Equation (19.5.32) can be expanded as follows:

$$S_N(\omega) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} X_k X_m e^{-j\omega(k-m)} \tag{19.5.33}$$

Substituting $h = (k-m)$ and $j = m$ in Eq. (19.5.33), the summation is carried out along the diagonal with limits found by referring to Fig. 19.4.1, resulting in

$$\begin{aligned} S_N(\omega) = &\sum_{h=0}^{N-1}\left[\frac{1}{N}\sum_{m=-[(N-1)/2]+h}^{(N-1)/2} X_m X_{m+h}\right]e^{-j\omega h} \\ &+ \sum_{h=-(N-1)}^{0}\left[\frac{1}{N}\sum_{m=-[(N-1)/2]}^{[(N-1)/2]-|h|} X_m X_{m+h}\right]e^{-j\omega h} \end{aligned} \tag{19.5.34}$$

The estimated autocorrelation $\hat{R}_X(h)$ from Eq. (19.4.76) is

$$\hat{R}_X(h) = \frac{1}{N}\sum_{i=0}^{N-h-1} X_i X_{i+h} \tag{19.5.35}$$

and substituting Eq. (19.5.35) in Eq. (19.5.34), we obtain

$$S_N(\omega) = \sum_{h=0}^{N-1} \hat{R}_X(h)e^{-j\omega h} + \sum_{h=-(N-1)}^{0} \hat{R}_X(h)e^{-j\omega h}$$

$$= \sum_{h=-(N-1)}^{N-1} \hat{R}_X(h)e^{-j\omega h} \qquad (19.5.36)$$

Taking expectations and substituting Eq. (19.4.76b) in Eq. (19.5.36), we obtain

$$E[S_N(\omega)] = \sum_{h=-(N-1)}^{N-1} E[\hat{R}_X(h)]e^{-j\omega h} = \sum_{h=-(N-1)}^{N-1} \left[1 - \frac{|h|}{N}\right] R_X(h)e^{-j\omega h} \qquad (19.5.37)$$

Taking the limit as of Eq. (19.5.37) as $N \to \infty$, we have

$$\lim_{N\to\infty} E[S_N(\omega)] = \lim_{N\to\infty} \sum_{h=-(N-1)}^{N-1} \left[1 - \frac{|h|}{N}\right] R_X(h)e^{-j\omega h}$$

$$= \lim_{N\to\infty} \text{DTFT}\left[R_X(h)\left(1 - \frac{|h|}{N}\right)\right] \qquad (19.5.38)$$

We use the frequency convolution property of the DTFT

$$x_n y_n \Longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(p)Y(\omega - p)dp$$

with $R_X(h) \Leftrightarrow S_X(\omega)$ and from Example 19.5.10

$$\left(1 - \frac{|h|}{N}\right) \Longleftrightarrow \frac{1}{N}\left[\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right]^2$$

Eq. (19.5.38) can be written as

$$\lim_{N\to\infty} E[S_N(\omega)] = \lim_{N\to\infty} \text{DTFT}\left[R_X(h)\left(1 - \frac{|h|}{N}\right)\right]$$

$$= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(p) \frac{1}{N}\left[\frac{\sin[(\omega - p)N/2]}{\sin[(\omega - p)/2]}\right] dp \qquad (19.5.39)$$

Using the result

$$\lim_{N\to\infty} \frac{1}{N}\left[\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right] \longrightarrow \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi)$$

in Eq. (19.5.39), we obtain

$$\lim_{N\to\infty} E[S_N(\omega)] = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(p)\delta(\omega - p)dp = S_X(\omega) \qquad (19.5.40)$$

Thus, the periodogram $S_N(\omega)$ is an asymptotically unbiased estimator of the psd $S_X(\omega)$. The variance of $S_N(\omega)$ can be obtained by using techniques similar to the continuous case, and we can show a result similar to Eq. (19.5.28):

$$\lim_{N\to\infty} \text{var}[S_N(\omega)] \approx S_X^2(\omega), \quad \omega \neq 0 \tag{19.5.41}$$

This equation shows that the estimator $S_N(\omega)$ is not a consistent estimator of $S_X(\omega)$.

Example 19.5.12 To check the validity of Eq. (19.5.41), a computer simulation of discrete white noise of zero mean and unit variance was performed. The power spectral density of the white noise is the variance, or $S_X(\omega) = \sigma_X^2 = 1$. The number 2^m of data points with $m = 7,9,10,11$ were chosen so that they could fit into the discrete Fourier transform algorithm. The psd $S(\omega)$ was estimated for $N = 128$, 512, 1024, and 2048 points using Eq. (19.5.32). Because of the symmetry about $N/2$, the estimates $S_{128}(\omega)$,

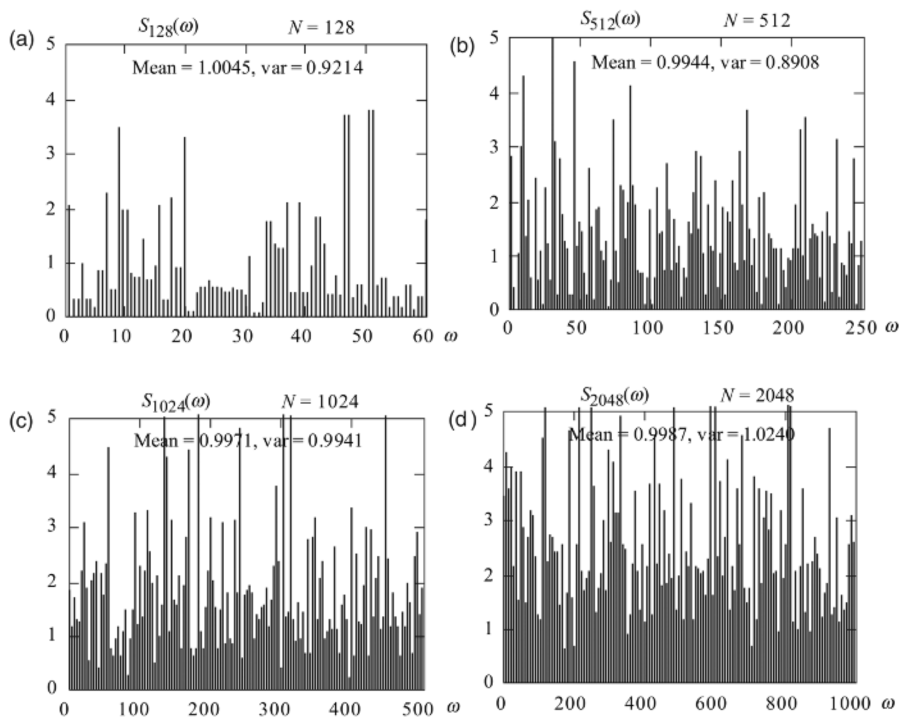

FIGURE 19.5.15

TABLE 19.5.1.

#	PSD	N	Mean	Variance
1	$S_X(\omega)$	—	1	0
2	$S_{128}(\omega)$	128	1.0045	0.9214
3	$S_{512}(\omega)$	512	0.9944	0.8908
4	$S_{1024}(\omega)$	1024	0.9971	0.9941
5	$S_{2048}(\omega)$	2048	0.9987	1.0240

$S_{512}(\omega)$, $S_{1024}(\omega)$, and $S_{2048}(\omega)$ are graphed for only half the number of data points in Figs. 19.5.15a–19.5.15d.

The estimated psd's and their variances are shown in Table 19.5.1.

The table shows that the estimator $S_N(\omega)$ is unbiased because the means for all N are nearly equal to 1 as expected. However, the variances of the psd for all N are nearly the same, equaling the derived value $\sigma_X^4 = 1$, which is the expected result according to Eq. (19.5.41). Thus, the simulation of discrete white noise confirms the desired theoretical result.