Chapter 4
Poisson Processes

The introduction to stochastic processes begins with a relatively simply type of process called a Poisson process that is essentially a type of counting process. For example as mentioned in Chap. 1, Bortkiewicz found 1898 that the number of deaths due to horse kicks in the Prussian army could be described by a Poisson random variable. A model depicting the cumulative number of deaths over time in the Prussian army would thus be described as a Poisson process. This chapter gives the formal description of such processes and includes some common extensions.

4.1 Basic Definitions

As stated in the first chapter, a random variable is simply a function that assigns a real number to each possible outcome in the sample space. However, we are usually interested in more than just a single random variable. For example, the daily demand for a particular product during a year would form a sequence of 365 random variables, all of which may influence a single decision. For this reason we usually consider sequences of random variables and their properties.

Definition 4.1. A *stochastic process* is a sequence of random variables. □

It is possible for a stochastic process to consist of a countable number of random variables, like the sequence of daily temperatures, in which case it is called a *discrete parameter process*. It is also possible that a stochastic process consists of an uncountable number of random variables in which case it is called a *continuous parameter process*. An example of a continuous parameter process is the continuous monitoring of the working condition of a machine. That is, at every point in time during the day a random variable is defined which designates the machine's working condition by setting the random variable equal to one if the machine is working and equal to zero if the machine is not working. For example, if the machine was working 55.78 minutes after the day had started, then $X_{55.78} = 1$. Thus, this stochastic process would be represented using a continuous parameter as $\{X_t\ ; t \geq 0\}$.

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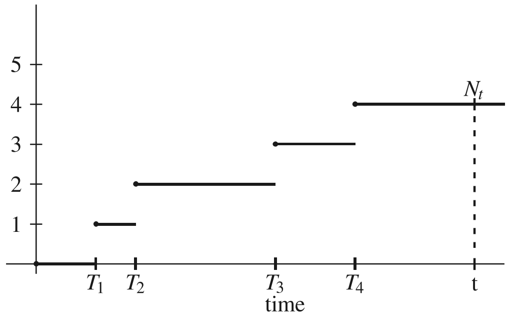

Fig. 4.1 Typical realization for a Poisson process, $\{N_t\,;t \geq 0\}$, with $T_n, n = 1, \cdots$, denoting arrival times

Definition 4.2. The set of real numbers containing the ranges of all the random variables in a stochastic process is called the *state space* of the stochastic process. □

The state space of a process may be either discrete or continuous. The continuous parameter stochastic process mentioned above that represented the continuous monitoring of a machine had a discrete state space of $E = \{0, 1\}$.

Example 4.1. We are interested in the dynamics of the arrival of telephone calls to a call center. To describe these arriving calls, consider a collection of random variables $\{N_t\ ;t \geq 0\}$, where each random variable N_t, for a fixed t, denotes the cumulative number of calls coming into the center by time t. Since these calls record a count, the state space is the set of whole numbers $E = \{0, 1, 2, \cdots\}$. In other words, the arrival of telephone calls can be modeled as a continuous parameter stochastic process with a countable state space. □

The arrival process described in the above example is typical of the type of processes introduced in this chapter, namely Poisson processes (see Fig. 4.1). As these are discussed, the terminology of an arrival process will be used generically. For example, a Poisson process may be used to describe the occurrence of errors during the transmission of an electronic message and when an error occurs, it would be described as an "arrival" to the process.

Definition 4.3. Let the continuous parameter stochastic process $\{N_t\ ;t \geq 0\}$ with $N_0 = 0$ have a state space equal to the nonnegative integers. It is called a *Poisson process* with rate λ if

(a) $\Pr\{N_t = k\} = \mathrm{e}^{-\lambda t}\,(\lambda t)^k/k!$ for all nonnegative k and $t \geq 0$.
(b) The event $\{N_{s+u} - N_s = i\}$ is independent of the event $\{N_t = j\}$ if $t < s$.
(c) The probability $\Pr\{N_{s+u} - N_s = i\}$ only depends on the value of u.

□

Condition (a) of Definition 4.3 is the defining characteristic of the Poisson process and is the reason for the name of the process. The characteristic specified in Condition (b) indicates that a Poisson process has *independent increments*. In other words, if you consider two non-overlapping intervals, what happens in one interval does not effect what happens in the second interval. The characteristic of Condition

(c) indicates that a Poisson process has *stationary increments*. In other words, the probability that a specified number of arrivals occurs within an interval only depends on the size of the interval not the location of interval.

For example, if the process of Example 4.1 has independent increments, then knowledge of the number of calls that arrived before 11:00AM will not help in predicting the number of calls to arrive between 11:00AM and noon. Likewise, if the process has stationary increments, then the probability that k calls arrive between 10:00AM and 10:15AM will be equal to the probability that k calls arrive between 4:00PM and 4:15PM because the size of both intervals is the same. It turns out that an arrival process with independent and stationary increments in which arrivals can only occur one-at-a-time must be a Poisson process (see [1]) so in that sense Condition (a) is redundant; however, because it is the characteristic that gives the process its name, it seems more descriptive to include it.

Using the definition of a Poisson random variable (Eq. 1.12), it is clear that

$$E[N_t] = \lambda\, t\,. \tag{4.1}$$

In other words, λ gives the mean arrival rate per unit time for an arrival process that is described by a Poisson process with rate λ. Also, because a Poisson process has independent and stationary increments, we also have, for nonnegative k

$$\Pr\{N_{s+u} - N_s = k\} = \frac{\mathrm{e}^{-\lambda u}\,(\lambda u)^k}{k!} \quad \text{for } u \geq 0\,. \tag{4.2}$$

Example 4.2. Assume that the arrivals to the call center described in Example 4.1 can be modeled according to a Poisson process with rate 25/hr. The number of calls that are expected within an eight-hour shift is $E[N_{t=8}] = 25 \times 8 = 200$. Assume we had 20 calls that arrived from 8AM through 9AM. What is the probability that there will be no calls that arrive from 9:00AM through 9:06AM? Because of independent increments, the information regarding the 20 calls is irrelevant. Because of stationary increments, we only need to know that the length of the interval is 6 minutes (or 0.1 hours), note that it starts at 9AM; thus, the answer is given as

$$\Pr\{N_{0.1} = 0\} = \frac{\mathrm{e}^{-25\times 0.1}\,(25\times 0.1)^0}{0!} = \mathrm{e}^{-2.5} = 0.08208\,.$$

We are also interested in the probability that more than 225 calls arrive during the eight-hour shift. To answer this question, observe that the normal is a good approximation for a Poisson random variable with a large mean. For the approximation, we let X be a normally distributed random variable with the same mean and variance as the Poisson random variable; thus,

$$\begin{aligned}\Pr\{N_8 > 225\} &\approx \Pr\{X > 225.5\}\\ &= \Pr\{Z > (225.5 - 200)/\sqrt{200}\}\\ &= \Pr\{Z > 1.80\} = 1 - 0.9641 = 0.0359\,.\end{aligned}$$

(For review, see the discussion regarding the use of a continuous distribution to approximate a discrete random variable on p. 21 and Example 1.10.) □

- *Suggestion: Do Problem 4.1.*

4.2 Properties and Computations

Because of the independent and stationary increment property of a Poisson process, the distribution of inter-arrival times can be easily determined. Consider an interval of length t. The probability that no arrival occurs within that interval is given by

$$\Pr\{N_t = 0\} = \mathrm{e}^{-\lambda t} .$$

Let T denote the time that the first arrival occurs. The event $\{T > t\}$ is equivalent to the event $\{N_t = 0\}$; therefore,

$$\Pr\{T \leq 0\} = 1 - \Pr\{T > 0\} = 1 - \mathrm{e}^{-\lambda t} \quad \text{for } t \geq 0$$

which is the CDF for the exponential distribution. Extending this by taking advantage of independent and stationary increments, it is not too difficult to show that a Poisson process always has exponential times between arrivals. As mentioned in the first chapter, the exponential distribution has no memory (see Problem 1.23). Using this fact, it is possible to show that a process that has exponential times between arrivals will have independent and stationary increments and thus form a Poisson process.

Property 4.1. *A Poisson process with rate λ has exponentially distributed inter-arrival times with the mean time between arrivals being $1/\lambda$. The converse is also true; namely, an arrival process with exponentially distributed inter-arrival times is a Poisson process.*

Example 4.3. Concern has been expressed regarding the occurrence of accidents at a relatively busy intersection. The time between accidents has been analyzed and it has been determined that this time is random and follows the exponential distribution with a mean time of five days between accidents. It is now noon on Monday and an accident has just occured and we would like to know the probability that the next accident will occur within the next 48 hours. This is given by

$$\Pr\{T_1 \leq 2\} = 1 - \mathrm{e}^{-2/5} = 0.3297 .$$

Let us say that it is now noon on Wednesday and no accident happened since Monday and we would like to know what is the probability that the next accident will occur within the next 48 hours. Due to the lack of memory of the exponential, the calculations are exactly the same as above.

Our final calculation for this example is to determine the probability that there will be at least four accidents next week. To determine this, we first observe that the occurrence of accidents form a Poisson process with rate $\lambda = 0.2$ days since the inter-accident times were exponential. (Note that mean rates and mean times are reciprocals of each other.) Since a week is seven days, we have that $\lambda t = 1.4$ so the solution is given by

$$\begin{aligned}
\Pr\{N_7 \geq 4\} &= 1 - \Pr\{N_7 \leq 3\} \\
&= 1 - \left(e^{1.4} + 1.4\,e^{1.4} + \frac{(1.4)^2\,e^{1.4}}{2} + \frac{(1.4)^3\,e^{1.4}}{3!} \right) \\
&= 1 - (0.2466 + 0.3452 + 0.2417 + 0.1128) = 0.0537\,.
\end{aligned}$$

□

Consider a Poisson arrival process with rate λ and denote the successive arrival times by $T_1, T_2, \cdots$. Based on Property 4.1, the time of the n^{th} arrival is composed of n exponential inter-arrival times which implies that the n^{th} arrival time is an Erlang Type-n distribution. Since each inter-arrival time has a mean of $1/\lambda$, the n^{th} arrival time has a mean of n/λ. Using this with Eq. (1.16) gives the following property.

Property 4.2. *Let T_n denote the n^{th} arrival time for a Poisson process with rate λ. The random variable T_n has an Erlang distribution with pdf given by*

$$f(t) = \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} \quad \textit{for } t \geq 0\,.$$

Assume that we are interested in whether or not T_n has occured; in other words, we would like to evaluate $\Pr\{T_n \leq t\}$ for some fixed value of n and t. In general, to answer a probability questions of a continuous random variable, the pdf must be integrated; however, in this case, it may be easier to sum over a range of discrete probabilities. Note that the event $\{T_n \leq t\}$ is equivalent to the event $\{N_t \geq n\}$; in other words, the n^{th} arrival occurs before time t if and only if there were n or more arrivals that occured in the time interval $[0,t]$. With the equivalence of these events, we can switch from a probability statement dealing with a continuous random variable to a probability statement dealing with a discrete random variable; thus

$$\Pr\{T_n \leq t\} = \Pr\{N_t \geq n\} = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,, \tag{4.3}$$

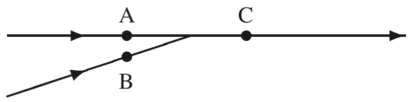

Fig. 4.2 If streams passing Points A and B are Poisson, then the merged stream passing Point C is Poisson as in Example 4.4

where T_n is the n^{th} arrival for a Poisson process with rate λ. For example, the calculations shown in Example 4.3 was to determine $\Pr\{N_7 \geq 4\}$. The same calculations also could be used to find $\Pr\{T_4 \leq 7\}$.

Before leaving the basics of Poisson processes, we mention two other properties that are convenient for modeling purposes; namely Poisson processes maintain their Poisson nature under both superposition and decomposition.

Property 4.3. ***Superposition of Poisson Processes:*** *Let $\{N_t\ ; t \geq 0\}$ and $\{M_t\ ; t \geq 0\}$ be two independent Poisson processes with rates λ_1 and λ_2. Form a third process by $Y_t = N_t + M_t$ for each $t \geq 0$. The process $\{Y_t\ ; t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.*

Example 4.4. Consider the diagram in Fig. 4.2 representing a limited access highway. Assume that the times in which cars pass Point A form a Poisson process with rate $\lambda_1 = 8$/min and cars passing Point B form a Poisson process with rate $\lambda_2 = 2$/min. After these two streams of cars merge, the times at which cars pass Point C form a Poisson process with mean rate 10/min. □

Property 4.4. ***Decomposition of a Poisson Processes:*** *Let $N = \{N_t\ ; t \geq 0\}$ be a Poisson processes with rate λ and let $\{X_1, X_2, \cdots\}$ denote an i.i.d. sequence of Bernoulli random variables (Eq. 1.9) independent of the Poisson process such that $\Pr\{X_n = 1\} = p$. Let $M = \{M_t\ ; t \geq 0\}$ be a new process formed as follows: for each positive n, consider the n^{th} arrival to the N process to also be an arrival to the M process if $X_n = 1$; otherwise, the n^{th} arrival to the N process is not part of the M process. The resulting M process is a Poisson process with rate λp.*

Example 4.5. Consider traffic coming to a fork in the road, and assume that the arrival times of cars to the fork form a Poisson process with mean rate 2 per minute. In addition, there is a 30% chance that cars will turn left and a 70% chance that cars will turn right. Under the assumption that all cars act independently, the arrival stream on the left-hand fork form a Poisson process with rate 0.6/min and the stream on the right-hand fork form a Poisson process with rate 1.4/min. □

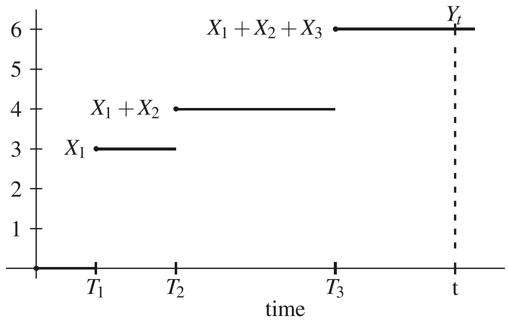

Fig. 4.3 Typical realization for a compound Poisson process, $\{Y_t\,; t \geq 0\}$, with $T_n, n = 1, \cdots$, denoting arrival times and $X_n, n = 1, \cdots$, batch sizes

- *Suggestion: Do Problems 4.2–4.5.*

4.3 Extensions of a Poisson Process

There are two common extensions to the Poisson process that are used for modeling purposes. The first extension is to allow more than one arrival at a specific point in time, and the second is to allow the rate to vary with respect to time.

4.3.1 Compound Poisson Processes

Implicit in Condition (a) of Definition 4.3 for Poisson processes is that arrivals occur one-at-a-time; however, there are many processes for which this assumption needs to be relaxed. Towards this end, consider a Poisson arrival process in which the arrivals are batches of items and the batch sizes form a sequence of independent and identically distributed (*i.i.d.*) positive random variables. The arrival of individual items then form a *compound Poisson process* as illustrated in Fig. 4.3.

Definition 4.4. Let $N = \{N_t\,; t \geq 0\}$ be Poisson process and let $\{X_1, X_2, \cdots\}$ be an *i.i.d.* sequence of random variables independent of N denoting the batch sizes of arrival. Define the process $\{Y_t\,; t \geq 0\}$ by

$$Y_t = \begin{cases} 0 & \text{if } N_t = 0 \\ \sum_{k=0}^{N_t} X_k & \text{if } N_t = 1, 2, \cdots . \end{cases}$$

The process $\{Y_t\,; t \geq 0\}$ is called a *compound Poisson process.* □

Even though the random variables X_n for $n = 1, \cdots$, are described as the batch sizes of the arriving batches, the values of X_n need not be integer and, in fact, may even be negative. For example batch sizes could refer to the amount of water added to a reservoir or removed (a negative quantity) from the reservoir. We can take ad-