Chapter 2
Poisson Processes

2.1 Exponential Distribution

To prepare for our discussion of the Poisson process, we need to recall the definition and some of the basic properties of the exponential distribution. A random variable T is said to have **an exponential distribution with rate** λ, or $T =$ exponential(λ), if

$$P(T \le t) = 1 - e^{-\lambda t} \quad \text{for all } t \ge 0 \tag{2.1}$$

Here we have described the distribution by giving the **distribution function** $F(t) = P(T \le t)$. We can also write the definition in terms of the **density function** $f_T(t)$ which is the derivative of the distribution function.

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{2.2}$$

Integrating by parts with $f(t) = t$ and $g'(t) = \lambda e^{-\lambda t}$,

$$\begin{aligned} ET &= \int t\, f_T(t)\, dt = \int_0^\infty t \cdot \lambda e^{-\lambda t}\, dt \\ &= -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\, dt = 1/\lambda \end{aligned} \tag{2.3}$$

Integrating by parts with $f(t) = t^2$ and $g'(t) = \lambda e^{-\lambda t}$, we see that

$$\begin{aligned} ET^2 &= \int t^2\, f_T(t)\, dt = \int_0^\infty t^2 \cdot \lambda e^{-\lambda t}\, dt \\ &= -t^2 e^{-\lambda t}\Big|_0^\infty + \int_0^\infty 2te^{-\lambda t}\, dt = 2/\lambda^2 \end{aligned} \tag{2.4}$$

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by the formula for ET. So the variance

$$\text{var}\,(T) = ET^2 - (ET)^2 = 1/\lambda^2 \tag{2.5}$$

While calculus is required to know the exact values of the mean and variance, it is easy to see how they depend on λ. Let $T = \text{exponential}(\lambda)$, i.e., have an exponential distribution with rate λ, and let $S = \text{exponential}(1)$. To see that S/λ has the same distribution as T, we use (2.1) to conclude

$$P(S/\lambda \leq t) = P(S \leq \lambda t) = 1 - e^{-\lambda t} = P(T \leq t)$$

Recalling that if c is any number then $E(cX) = cEX$ and $\text{var}\,(cX) = c^2\,\text{var}\,(X)$, we see that

$$ET = ES/\lambda \qquad \text{var}\,(T) = \text{var}\,(S)/\lambda^2$$

Lack of memory property. It is traditional to formulate this property in terms of waiting for an unreliable bus driver. In words, "if we've been waiting for t units of time then the probability we must wait s more units of time is the same as if we haven't waited at all." In symbols

$$P(T > t + s | T > t) = P(T > s) \tag{2.6}$$

To prove this we recall that if $B \subset A$, then $P(B|A) = P(B)/P(A)$, so

$$P(T > t + s | T > t) = \frac{P(T > t + s)}{P(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(T > s)$$

where in the third step we have used the fact $e^{a+b} = e^a e^b$.

Exponential races. Let $S = \text{exponential}(\lambda)$ and $T = \text{exponential}(\mu)$ be independent. In order for the minimum of S and T to be larger than t, each of S and T must be larger than t. Using this and independence we have

$$\begin{aligned} P(\min(S, T) > t) &= P(S > t, T > t) = P(S > t)P(T > t) \\ &= e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t} \end{aligned} \tag{2.7}$$

That is, $\min(S, T)$ has an exponential distribution with rate $\lambda + \mu$. The last calculation extends easily to a sequence of independent random variables $T_1, \ldots, T_n$ where $T_i = \text{exponential}(\lambda_i)$.

$$\begin{aligned} P(\min(T_1, \ldots, T_n) > t) &= P(T_1 > t, \ldots T_n > t) \\ &= \prod_{i=1}^{n} P(T_i > t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t} \end{aligned} \tag{2.8}$$

That is, the minimum, $\min(T_1, \ldots, T_n)$, of several independent exponentials has an exponential distribution with rate equal to the sum of the rates $\lambda_1 + \cdots \lambda_n$.

In the last paragraph we have computed the duration of a race between exponentially distributed random variables. We will now consider: "Who finishes first?" Going back to the case of two random variables, we break things down according to the value of S and then using independence with our formulas (2.1) and (2.2) for the distribution and density functions, to conclude

$$\begin{aligned} P(S < T) &= \int_0^\infty f_S(s) P(T > s)\, ds \\ &= \int_0^\infty \lambda e^{-\lambda s} e^{-\mu s}\, ds \\ &= \frac{\lambda}{\lambda + \mu} \int_0^\infty (\lambda + \mu) e^{-(\lambda+\mu)s}\, ds = \frac{\lambda}{\lambda + \mu} \end{aligned} \tag{2.9}$$

where on the last line we have used the fact that $(\lambda+\mu)e^{-(\lambda+\mu)s}$ is a density function and hence must integrate to 1.

From the calculation for two random variables, you should be able to guess that if $T_1, \ldots, T_n$ are independent exponentials, then

$$P(T_i = \min(T_1, \ldots, T_n)) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \tag{2.10}$$

That is, the probability of i finishing first is proportional to its rate.

Proof. Let $S = T_i$ and U be the minimum of T_j, $j \neq i$. (2.8) implies that U is exponential with parameter

$$\mu = (\lambda_1 + \cdots + \lambda_n) - \lambda_i$$

so using the result for two random variables

$$P(T_i = \min(T_1, \ldots, T_n)) = P(S < U) = \frac{\lambda_i}{\lambda_i + \mu} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

proves the desired result. □

Let I be the (random) index of the T_i that is smallest. In symbols,

$$P(I = i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

You might think that the T_i's with larger rates might be more likely to win early. However,

$$I \text{ and } V = \min\{T_1, \ldots T_n\} \text{ are independent.} \tag{2.11}$$

Proof. Let $f_{i,V}(t)$ be the density function for V on the set $I = i$. In order for i to be first at time t, $T_i = t$ and the other $T_j > t$ so

$$\begin{aligned} f_{i,V}(t) &= \lambda_i e^{-\lambda_i t} \cdot \prod_{j \neq i} e^{-\lambda_j t} \\ &= \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \cdot (\lambda_1 + \cdots + \lambda_n) e^{-(\lambda_1 + \cdot + \lambda_n)t} \\ &= P(I = i) \cdot f_V(t) \end{aligned}$$

since V has an exponential$(\lambda_1 + \cdots + \lambda_n)$ distribution. □

Our final fact in this section concerns sums of exponentials.

Theorem 2.1. *Let $\tau_1, \tau_2, \ldots$ be independent exponential(λ). The sum $T_n = \tau_1 + \cdots + \tau_n$ has a gamma(n, λ) distribution. That is, the density function of T_n is given by*

$$f_{T_n}(t) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \textit{for } t \geq 0 \tag{2.12}$$

and 0 otherwise.

Proof. The proof is by induction on n. When $n = 1$, T_1 has an exponential(λ) distribution. Recalling that the 0th power of any positive number is 1, and by convention we set $0! = 1$, the formula reduces to

$$f_{T_1}(t) = \lambda e^{-\lambda t}$$

and we have shown that our formula is correct for $n = 1$.

To do the induction step, suppose that the formula is true for n. The sum $T_{n+1} = T_n + \tau_{n+1}$, so breaking things down according to the value of T_n, and using the independence of T_n and t_{n+1}, we have

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(s) f_{t_{n+1}}(t-s)\, ds$$

Plugging the formula from (2.12) in for the first term and the exponential density in for the second and using the fact that $e^a e^b = e^{a+b}$ with $a = -\lambda s$ and $b = -\lambda(t-s)$ gives

$$\begin{aligned} \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} \cdot \lambda e^{-\lambda(t-s)}\, ds &= e^{-\lambda t} \lambda^n \int_0^t \frac{s^{n-1}}{(n-1)!}\, ds \\ &= \lambda e^{-\lambda t} \frac{\lambda^n t^n}{n!} \end{aligned}$$

which completes the proof. □

2.2 Defining the Poisson Process

In this section we will give two definitions of the **Poisson process with rate** λ. The first, which will be our official definition, is nice because it allows us to construct the process easily.

Definition. Let $\tau_1, \tau_2, \ldots$ be independent exponential(λ) random variables. Let $T_n = \tau_1 + \cdots + \tau_n$ for $n \geq 1, T_0 = 0$, and define $N(s) = \max\{n : T_n \leq s\}$.

We think of the τ_n as times between arrivals of customers at a bank, so $T_n = \tau_1 + \cdots + \tau_n$ is the arrival time of the nth customer, and $N(s)$ is the number of arrivals by time s. To check the last interpretation, consider the following example: and note that $N(s) = 4$ when $T_4 \leq s < T_5$, that is, the fourth customer has arrived by time s but the fifth has not.

Recall that X has a **Poisson distribution** with mean λ, or $X = \text{Poisson}(\lambda)$, for short, if

$$P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots$$

To explain why $N(s)$ is called the Poisson process rather than the exponential process, we will compute the distribution of $N(s)$.

Lemma 2.2. *$N(s)$ has a Poisson distribution with mean λs.*

Proof. Now $N(s) = n$ if and only if $T_n \leq s < T_{n+1}$; i.e., the nth customer arrives before time s but the $(n+1)$th after s. Breaking things down according to the value of $T_n = t$ and noting that for $T_{n+1} > s$, we must have $\tau_{n+1} > s - t$, and τ_{n+1} is independent of T_n, it follows that

$$P(N(s) = n) = \int_0^s f_{T_n}(t) P(t_{n+1} > s - t)\, dt$$

Plugging in (2.12) now, the last expression is

$$= \int_0^s \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda(s-t)}\, dt$$

$$= \frac{\lambda^n}{(n-1)!} e^{-\lambda s} \int_0^s t^{n-1}\, dt = e^{-\lambda s} \frac{(\lambda s)^n}{n!}$$

which proves the desired result. □

Since this is our first mention of the Poisson distribution, we pause to derive some of its properties.

Theorem 2.3. *For any $k \geq 1$*

$$EX(X-1)\cdots(X-k+1) = \lambda^k \tag{2.13}$$

and hence $var(X) = \lambda$

Proof. $X(X-1)\cdots(X-k+1) = 0$ if $X \le k-1$ so

$$EX(X-1)\cdots(X-k+1) = \sum_{j=k}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!} j(j-1)\cdots(j-k+1)$$
$$= \lambda^k \sum_{j=k}^{\infty} e^{-\lambda}\frac{\lambda^{j-k}}{(j-k)!} = \lambda^k$$

since the sum gives the total mass of the Poisson distribution. Using $\text{var}(X) = E(X(X-1)) + EX - (EX)^2$ we conclude

$$\text{var}(X) = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

□

Theorem 2.4. *If X_i are independent Poissson(λ_i) then*

$$X_1 + \cdots + X_k = \textit{Poisson}(\lambda_1 + \cdots + \lambda_n).$$

Proof. It suffices to prove the result for $k = 2$, for then the general result follows by induction.

$$P(X_1 + X_2 = n) = \sum_{m=0}^{n} P(X_1 = m)P(X_2 = n-m)$$
$$= \sum_{m=0}^{n} e^{-\lambda_1}\frac{(\lambda_1)^m}{m!} \cdot e^{-\lambda_2}\frac{(\lambda_2)^{n-m}}{(n-m)!}$$

Knowing the answer we want, we can rewrite the last expression as

$$e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!} \cdot \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

The sum is 1, since it is the sum of all the probabilities for a binomial(n, p) distribution with $p = \lambda_1/(\lambda_1+\lambda_2)$. The term outside the sum is the desired Poisson probability, so have proved the desired result. □

The property of the Poisson process in Lemma 2.2 is the first part of our second definition. To start to develop the second part we prove a Markov property:

Lemma 2.5. *$N(t+s) - N(s), t \ge 0$ is a rate λ Poisson process and independent of $N(r), 0 \le r \le s$.*

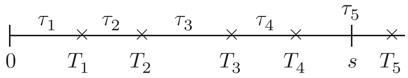

Fig. 2.1 Poisson process definitions

Why is this true? Suppose for concreteness (and so that we can use Fig. 2.1 at the beginning of this section again) that by time s there have been four arrivals T_1, T_2, T_3, T_4 that occurred at times t_1, t_2, t_3, t_4. We know that the waiting time for the fifth arrival must have $\tau_5 > s - t_4$, but by the lack of memory property of the exponential distribution (2.6)

$$P(\tau_5 > s - t_4 + t | \tau_5 > s - t_4) = P(\tau_5 > t) = e^{-\lambda t}$$

This shows that the distribution of the first arrival after s is exponential(λ) and independent of T_1, T_2, T_3, T_4. It is clear that $\tau_6, \tau_7, \ldots$ are independent of T_1, T_2, T_3, T_4, and τ_5. This shows that the interarrival times after s are independent exponential(λ), and hence that $N(t+s) - N(s), t \geq 0$ is a Poisson process. □

From Lemma 2.5 we get easily the following:

Lemma 2.6. *$N(t)$ has **independent increments**: if $t_0 < t_1 < \ldots < t_n$, then*

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots N(t_n) - N(t_{n-1}) \quad \textit{are independent}$$

Why is this true? Lemma 2.5 implies that $N(t_n) - N(t_{n-1})$ is independent of $N(r), r \leq t_{n-1}$ and hence of $N(t_{n-1}) - N(t_{n-2}), \ldots N(t_1) - N(t_0)$. The desired result now follows by induction. □

We are now ready for our second definition. It is in terms of the process $\{N(s) : s \geq 0\}$ that counts the number of arrivals in $[0, s]$.

Theorem 2.7. *If $\{N(s), s \geq 0\}$ is a Poisson process, then*

(i) $N(0) = 0$,
(ii) $N(t+s) - N(s) =$ *Poisson*(λt), *and*
(iii) $N(t)$ *has independent increments.*

Conversely, if (i), (ii), and (iii) hold, then $\{N(s), s \geq 0\}$ is a Poisson process.

Why is this true? Lemmas 2.2 and 2.6 prove (ii) and (iii). To start to prove the converse, let T_n be the time of the nth arrival. The first arrival occurs after time t if and only if there were no arrivals in $[0, t]$. So using the formula for the Poisson distribution

$$P(\tau_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

This shows that $\tau_1 = T_1$ is exponential(λ). For $\tau_2 = T_2 - T_1$ we note that

$$\begin{aligned} P(\tau_2 > t | \tau_1 = s) &= P(\text{ no arrival in } (s, s+t] \,|\tau_1 = s) \\ &= P(N(t+s) - N(s) = 0 | N(r) = 0 \text{ for } r < s, N(s) = 1) \\ &= P(N(t+s) - N(s) = 0) = e^{-\lambda t} \end{aligned}$$

by the independent increments property in (iii), so τ_2 is exponential(λ) and independent of τ_1. Repeating this argument we see that $\tau_1, \tau_2, \ldots$ are independent exponential(λ). □

Up to this point we have been concerned with the mechanics of defining the Poisson process, so the reader may be wondering:

Why is the poisson process important for applications? Our answer is based on the Poisson approximation to the binomial. Suppose that each of the n students on Duke campus flips coins with probability λ/n of heads to decide if they will go to the Great Hall (food court) between 12:17 and 12:18 . The probability that exactly k students will go during the 1-min time interval is given by the binomial($n, \lambda/n$) distribution

$$\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \tag{2.14}$$

Theorem 2.8. *If n is large the binomial($n, \lambda/n$) distribution is approximately Poisson(λ).*

Proof. Exchanging the numerators of the first two fractions and breaking the last term into two, (2.14) becomes

$$\frac{\lambda^k}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k}\cdot\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} \tag{2.15}$$

Considering the four terms separately, we have

(i) $\lambda^k/k!$ does not depend on n.

(ii) There are k terms on the top and k terms on the bottom, so we can write this fraction as

$$\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}$$

For any j we have $(n-j)/n \to 1$ as $n \to \infty$, so the second term converges to 1 as $n \to \infty$.

(iii) Skipping to the last term in (2.15), $\lambda/n \to 0$, so $1-\lambda/n \to 1$. The power $-k$ is fixed so

$$\left(1-\frac{\lambda}{n}\right)^{-k} \to 1^{-k} = 1$$

(iv) We broke off the last piece to make it easier to invoke one of the famous facts of calculus:

$$(1-\lambda/n)^n \to e^{-\lambda} \quad \text{as } n \to \infty.$$

If you haven't seen this before, recall that

$$\log(1-x) = -x + x^2/2 + \ldots$$

so we have $n\log(1-\lambda/n) = -\lambda + \lambda^2/n + \ldots \to \lambda$ as $n \to \infty$.

Combining (i)–(iv), we see that (2.15) converges to

$$\frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1$$

which is the Poisson distribution with mean λ. □

By extending the last argument we can also see why the number of individuals that arrive in two disjoint time intervals should be independent. Using the multinomial instead of the binomial, we see that the probability j people will go between 12:17 and 12:18 and k people will go between 12:31 and 12:33 is

$$\frac{n!}{j!k!(n-j-k)!} \left(\frac{\lambda}{n}\right)^j \left(\frac{2\lambda}{n}\right)^k \left(1 - \frac{3\lambda}{n}\right)^{n-(j+k)}$$

Rearranging gives

$$\frac{(\lambda)^j}{j!} \cdot \frac{(2\lambda)^k}{k!} \cdot \frac{n(n-1)\cdots(n-j-k+1)}{n^{j+k}} \cdot \left(1 - \frac{3\lambda}{n}\right)^{n-(j+k)}$$

Reasoning as before shows that when n is large, this is approximately

$$\frac{(\lambda)^j}{j!} \cdot \frac{(2\lambda)^k}{k!} \cdot 1 \cdot e^{-3\lambda}$$

Writing $e^{-\lambda} = e^{-\lambda/3} e^{-2\lambda/3}$ and rearranging we can write the last expression as

$$e^{-\lambda} \frac{\lambda^j}{j!} \cdot e^{-2\lambda} \frac{(2\lambda)^k}{k!}$$

This shows that the number of arrivals in the two time intervals we chose are independent Poissons with means λ and 2λ.

The last proof can be easily generalized to show that if we divide the hour between 12:00 and 1:00 into any number of intervals, then the arrivals are independent Poissons with the right means. However, the argument gets very messy to write down.

More realistic models. Two of the weaknesses of the derivation above are:

(i) All students are assumed to have exactly the same probability of going to the Great Hall.
(ii) The probability of going in a given time interval is a constant multiple of the length of the interval, so the arrival rate of customers is constant during the hour. In reality there is a large influx of people between 11:30 and 11:45 soon after the end of 10:10–11:25 classes.

(i) is a very strong assumption but can be weakened by using a more general Poisson approximation result like the following:

Theorem 2.9. *Let* $X_{n,m}, 1 \le m \le n$ *be independent random variables with* $P(X_m = 1) = p_m$ *and* $P(X_m = 0) = 1 - p_m$. *Let*

$$S_n = X_1 + \cdots + X_n, \quad \lambda_n = ES_n = p_1 + \cdots + p_n,$$

and $Z_n = Poisson(\lambda_n)$. *Then for any set A*

$$|P(S_n \in A) - P(Z_n \in A)| \le \sum_{m=1}^{n} p_m^2$$

Why is this true? If X and Y are integer valued random variables then for any set A

$$|P(X \in A) - P(Y \in A)| \le \frac{1}{2} \sum_n |P(X = n) - P(Y = n)|$$

The right-hand side is called the **total variation distance** between the two distributions and is denoted $\|X - Y\|$. If $P(X = 1) = p$, $P(X = 0) = 1 - p$, and $Y = \text{Poisson}(p)$ then

$$\sum_n |P(X = n) - P(Y = n)| = |(1 - p) - e^{-p}| + |p - pe^{-p}| + 1 - (1 + p)e^{-p}$$

Since $1 \ge e^{-p} \ge 1 - p$ the right-hand side is

$$e^{-p} - 1 + p + p - pe^{-p} + 1 - e^{-p} - pe^{-p} = 2p(1 - e^{-p} \le 2p^2$$

Let $Y_m = \text{Poisson}(p_m)$ be independent. At this point we have shown $\|X_i - Y_i\| \le p_i^2$. With a little work one can show

$$\|(X_1 + \cdots + X_n) - (Y_1 + \cdots + Y_n)\|$$

$$\|(X_1, \cdots, X_n) - (Y_1, \cdots, Y_n)\| \le \sum_{m=1}^{n} \|X_m - Y_m\|$$

and the desired result follows. □

Theorem 2.9 is useful because it gives a bound on the difference between the distribution of S_n and the Poisson distribution with mean $\lambda_n = ES_n$. To bound the bound it is useful to note that

$$\sum_{m=1}^{n} p_m^2 \le \max_k p_k \left(\sum_{m=1}^{n} p_m \right)$$

so the approximation is good if $\max_k p_k$ is small. This is similar to the usual heuristic for the normal distribution: the sum is due to small contributions from a large number of variables. However, here small means that it is nonzero with small probability. When a contribution is made it is equal to 1.

The last results handles problem (i). To address the problem of varying arrival rates mentioned in (ii), we generalize the definition.

Nonhomogeneous Poisson processes. *We say that* $\{N(s), s \geq 0\}$ *is a Poisson process with rate* $\lambda(r)$ *if*

(i) $N(0) = 0$,
(ii) $N(t)$ *has independent increments, and*
(iii) $N(t) - N(s)$ *is Poisson with mean* $\int_s^t \lambda(r)\, dr$.

The first definition does not work well in this setting since the interarrival times $\tau_1, \tau_2, \ldots$ are no longer exponentially distributed or independent. To demonstrate the first claim, we note that

$$P(\tau_1 > t) = P(N(t) = 0) = e^{-\int_0^t \lambda(s)\, ds}$$

since $N(t)$ is Poisson with mean $\mu(t) = \int_0^t \lambda(s)\, ds$. Differentiating gives the density function

$$P(\tau_1 = t) = -\frac{d}{dt} P(t_1 > t) = \lambda(t) e^{-\int_0^t \lambda(s)\, ds} = \lambda(t) e^{-\mu(t)}$$

Generalizing the last computation shows that the joint distribution

$$f_{T_1,T_2}(u, v) = \lambda(u) e^{-\mu(u)} \cdot \lambda(v) e^{-(\mu(v) - \mu(u))}$$

Changing variables, $s = u, t = v - u$, the joint density

$$f_{\tau_1,\tau_2}(s, t) = \lambda(s) e^{-\mu(s)} \cdot \lambda(s + t) e^{-(\mu(s+t) - \mu(s))}$$

so τ_1 and τ_2 are not independent when $\lambda(s)$ is not constant.

2.3 Compound Poisson Processes

In this section we will embellish our Poisson process by associating an independent and identically distributed (i.i.d.) random variable Y_i with each arrival. By independent we mean that the Y_i are independent of each other and of the Poisson process of arrivals. To explain why we have chosen these assumptions, we begin with two examples for motivation.