Lesson 4

Poisson Processes

This Lesson is devoted entirely to an important class of *continuous-time* Markov chains, the Poisson processes. This Lesson also serves as an introduction to continuous-time Markov chains where the general theory will be treated in Lesson 5.

4.1 Motivation and Modeling

Consider a sequence of events which occur at random instants, say $T_1, T_2, \cdots, T_n, \cdots$. For example the arrival of customers for service; the occurrence of breakdowns, accidents, earthquakes; the arrivals of particles registered by a Geiger counter.

The sequence $(T_n,\, n \geq 1)$ is called a *point process.* In the following we will suppose that $0 < T_1 < T_2 < \cdots < T_n < \cdots$ and $\lim_{n\uparrow\infty} T_n = \infty$ with probability one. These properties mean that the registration of the events begins at time 0, that two events cannot occur at the same time and that the observed phenomena take place during a long period. Note that 0 is **not** an event time arrival. The reason is that it is natural to suppose that the distribution of T_n is continuous.

Now a convenient method for describing (T_n) is to consider the associated *counting process* $(N_t,\, t \geq 0)$ where N_t represents the number of events that have occured in the time interval $[0,\, t]$.

$(N_t,\, t \geq 0)$ and $(T_n, n = 1, 2, \cdots)$ contain the same information since with probability one

$$N_t = \sup\{n : n = 0, 1, 2, \cdots; T_n \leq t\}, t \geq 0 \tag{4.1}$$

with the conventional notation $T_0 = 0$; whereas

$$T_n = \inf\{t : t \geq 0,\ N_t \geq n\}; n = 0, 1, \cdots. \tag{4.2}$$

These relations are visible on figure 1 which shows a typical sample path for the Counting Process (N_t).

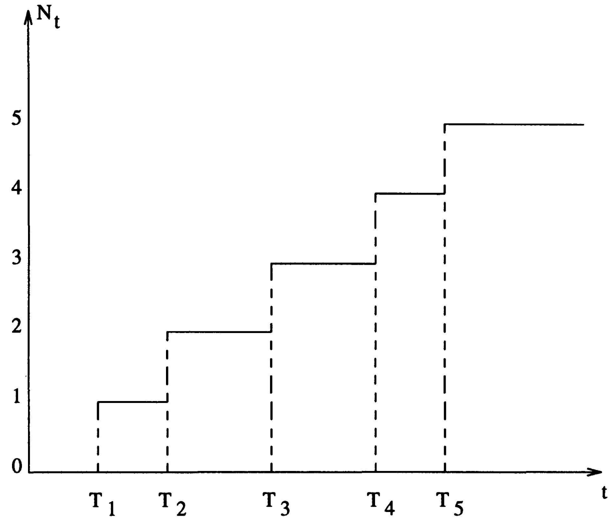

Figure 1. A typical sample path of a Counting Process

The following relations between (N_t) and (T_n) are also of interest

$$\{N_t = n\} = \{T_n \leq t \leq T_{n+1}\}, \tag{4.3}$$

$$\{N_t \geq n\} = \{T_n \leq t\}, \tag{4.4}$$

$$\{s < T_n \leq t\} = \{N_s < n \leq N_t\}. \tag{4.5}$$

On the other hand if the sources which generate the events are independent, then it is natural to suppose that the respective numbers of events which occur on nonoverlapping time intervals are stochastically independent.

Furthermore, if the sources keep the same intensity during the time then the distribution of $N_{t+h} - N_{s+h}$ does not depend on h.

4.2 Axioms of Poisson Processes

The above considerations lead to the following axioms
A_0: $0 < T_1 < T_2 < \cdots < T_n < \cdots$ and $\lim_{n \nearrow \infty} T_n = \infty$ with probability one.

A_1: $(N_t, t \geq 0)$ is an *independent increments* process, i.e., for any $k \geq 2$ and $0 \leq t_0 < t_1 < \cdots < t_k$ the random variables $N_{t_1} - N_{t_0}$, $N_{t_2} - N_{t_1}$, $\cdots$, $N_{t_k} - N_{t_{k-1}}$ are independent.

A_2: $(N_t, t \geq 0)$ is a *stationary increments* process, i.e., for any $h > 0$, $0 \leq s < t$, $N_{t+h} - N_{s+h}$ and $N_t - N_s$ have the same distribution.

If these axioms are valid we have the following astonishing result:

Theorem 4.1 *If A_0, A_1 and A_2 hold, then there exists a strictly positive constant λ such that, for each $0 \leq s < t$,*

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}; \qquad k = 0, 1, 2, \cdots. \tag{4.6}$$

Relation(4.6) means that $N_t - N_s$ follows the Poisson distribution with parameter $\lambda(t-s)$ (we use the notation $\mathcal{P}(\alpha)$ to denote a Poisson distribution with parameter α); λ is called the *intensity* of the Poisson process (N_t).

Note that (4.6) together with A_1 and A_2 determine completely the distribution of $(N_t, t \geq 0)$ since $N_0 = 0\, a.s.$ and since if $0 < t_1 < \cdots < t_k$

$$\begin{aligned} & P(N_{t_1} = n_1, \cdots, N_{t_k} = n_k) \\ = \; & P(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2 - n_1, \cdots, N_{t_k} - N_{t_{k-1}} = n_k - n_{k-1}), \end{aligned}$$

then using A_1, A_2 and (4.6) we obtain

$$\begin{aligned} & P(N_{t_1} = n_1, \cdots, N_{t_k} = n_k) = e^{-\lambda t_1} \frac{(\lambda t_1)^{n_1}}{n_1!} \cdots \\ \times \; & e^{-\lambda(t_k - t_{k-1})} \frac{(\lambda(t_k - t_{k-1}))^{n_k - n_{k-1}}}{(n_k - n_{k-1})!} 1_{0 \leq n_1 \leq \cdots \leq n_k}, \end{aligned} \tag{4.7}$$

where $n_1, \cdots, n_k \in \mathbb{N}$. Now, according to Kolmogorov's existence theorem (See Lesson 2), the distribution of the entire process is determined.

Before making some comments about the axioms, we give the proof of the Theorem 4.1. In that proof and in the following of the Lesson, the expression "with probability one" will be omitted.

Proof. Let g_{t-s} be the moment generating function of $N_t - N_s$:

$$g_{t-s}(u) = E\left(u^{N_t - N_s}\right) = \sum_{k=0}^{\infty} P(N_t - N_s = u)u^k, \qquad 0 \leq u \leq 1. \quad (4.8)$$

Using the decomposition $N_t = (N_t - N_s) + (N_s - N_0)$, and axioms A_1 and A_2 we get

$$g_t(u) = g_s(u) g_{t-s}(u), \quad 0 \leq s < t, \ 0 \leq u \leq 1, \quad (4.9)$$

which implies for each pair (p, q) of integers

$$g_{p/q}(u) = \left(g_{1/q}(u)\right)^p = \left((g_1(u))^{1/q}\right)^p = (g_1(u))^{p/q}. \quad (4.10)$$

On the other hand (4.9) entails the decrease of $t \longmapsto g_t(u)$, consequently (4.10) remains valid for irrational t's:

$$g_t(u) = (g_1(u))^t, \qquad t > 0. \quad (4.11)$$

We now show that $g_t(u)$ cannot vanish. In fact, if $g_{t_0}(u) = 0$ then (4.11) implies $g_1(u) = 0$ and consequently $g_t(u) = 0$ for each $t > 0$. This is a contradiction since

$$g_t(u) \geq P(N_t = 0) = P(T_1 > t) \uparrow P(T_1 > 0) = 1 \quad \text{as} \quad t \downarrow 0.$$

Finally we may let

$$g_t(u) = e^{-t\lambda(u)}, \quad (4.12)$$

where $\lambda(u)$ is positive.

It remains to identify $\lambda(u)$. To this aim we first show that

$$P(N_h \geq 2) = o(h) \quad \text{as} \quad h \to 0. \quad (4.13)$$

Note that for $h > 0$,

$$\sum_{n \geq 1} \{N_{(n-1)h} = 0, \ N_{nh} - N_{(n-1)h} \geq 2\} \subset \{T_2 < T_1 + h\}$$

then since $P(N_t = 0) = g_t(0) = e^{-t\lambda(0)}$, we obtain, using A_1 and A_2,

$$\sum_{n \geq 1} exp(-(n-1)h\lambda(0))\, P(N_h \geq 2) \leq P(T_2 < T_1 + h). \quad (4.14)$$

Now it is clear that $\lambda(0) \neq 0$, unless (4.12) implies $g_t(0) = 1$ for each $t > 0$, consequently $1 = P(N_t = 0) = P(T_1 > t)$ for each $t > 0$, hence

$T_1 = +\infty$ *a.s.* which contradicts A_0. Thus (4.14) may be written under the form

$$\frac{P(N_h \geq 2)}{1 - e^{-h\lambda(0)}} \leq P(T_2 < T_1 + h).$$

Now, as $h \downarrow 0$, $P(T_2 < T_1 + h) \downarrow P(T_2 \leq T_1) = 0$ and $1 - e^{-h\lambda(0)} \sim h\lambda(0)$ hence (4.13).

On the other hand, we have

$$\lambda(u) = \lim_{h \downarrow 0} \frac{1}{h}\left(1 - e^{-h\lambda(u)}\right)$$

so by (4.8) and (4.12)

$$\lambda(u) = \lim_{h \downarrow 0} \sum_{k \geq 1} \frac{1}{h} P(N_h = k)(1 - u^k).$$

Using (4.13) we obtain

$$0 \leq \lim_{h \downarrow 0} \sum_{k \geq 2} \frac{1}{h} P(N_h = k)(1 - u^k) \leq \lim_{h \downarrow 0} \frac{P(N_h \geq 2)}{h} = 0.$$

Consequently,

$$\lambda(u) = \lim_{h \downarrow 0} \frac{1}{h} P(N_h = 1)(1 - u) = \lambda(1 - u),$$

where $\lambda = \lim_{h \downarrow 0} P(N_h = 1)/h$. Finally

$$g_t(u) = e^{-\lambda t(1-u)}, \quad 0 \leq u \leq 1,$$

which is the moment generating function of $\mathcal{P}(\lambda t)$ and the proof is complete. ◇

The following important properties of (N_t) have been obtained in the above proof:

Corollary 4.1 *As* $h \to 0(+)$, *we have*

$$P(N_{t+h} - N_t = 0) = 1 - \lambda h + o(h), \tag{4.15}$$

$$P(N_{t+h} - N_t = 1) = \lambda h + o(h), \tag{4.16}$$

$$P(N_{t+h} - N_t \geq 2) = o(h). \tag{4.17}$$

Thus, for small h, $N_{t+h} - N_t$ follows approximately the Bernoulli distribution $\mathcal{B}(1, \lambda h)$: in a sufficiently small time interval, at most one event may occur and the probability of this occurrence is proportional to the length of that interval.

Comments about axioms

In order to construct Poisson Processes, other axioms may be used. Consider the axioms

A'_0: $N_0 = 0$; $0 < P(N_t > 0) < 1$, $t > 0$.

A_3: For any $t \geq 0$,

$$\lim_{h \to 0} \frac{P(N_{t+h} - N_t \geq 2)}{P(N_{t+h} - N_t = 1)} = 0.$$

Then A'_0 and A_3 together with A_1 and A_2 imply (4.6). Clearly A'_0 and A_3 are consequences of A_0, A_1, and A_2. It should be noticed that our axioms are simpler than classical systems like A'_0, A_1 , A_2, and A_3. The idea may be found in Neveu (1990).

4.3 Interarrival Times

Let $(N_t, t \geq 0)$ be a Counting Process associated with the Point Process $(T_n, n \geq 1)$. Set $T_0 = 0$ and consider the *interarrival times*

$$W_n = T_n - T_{n-1}, \qquad n \geq 1.$$

If (N_t) is a Poisson Process, then the sequence (W_n) has some special properties given by the following

Theorem 4.2 *Let (N_t) be a Poisson Process with intensity λ. Then the W_n's are independent with common exponential distribution characterized by*

$$P(W_n > t) = e^{-\lambda t}, \qquad t > 0,\ n \geq 1 \tag{4.18}$$

and consequently

$$E(W_n) = 1/\lambda, \qquad n \geq 1. \tag{4.19}$$

Theorem 4.2 contains an important and paradoxical property of Poisson Processes: if $n \geq 2$, W_n is the waiting time between two successive events, but this interpretation is not true for $W_1 = T_1 - T_0$ since $T_0 = 0$ is not an event time-arrival. However W_1 and W_n have the same distribution!

Proof of Theorem 4.2. It suffices to show that

$$P(W_1 > t_1, \cdots, W_n > t_n) = \prod_{i=1}^{n} e^{-\lambda t_i}, \; t_1, \cdots, t_n \geq 0; \; n \geq 1. \qquad (4.20)$$

If $n = 1$ the result follows from Theorem 4.1 since

$$P(W_1 > t_1) = P(T_1 > t_1) = P(N_t = 0) = e^{-\lambda t_1}.$$

Now for convenience, we only establish (4.18) for $n = 2$. A similar proof could be given for $n > 2$.

Taking $0 \leq s_1 < t_1 < s_2 < t_2$, we may write

$$\begin{aligned} & P(s_1 < T_1 < t_1, s_2 < T_2 < t_2) \\ = \; & P(N_{s_1} = 0, N_{t_1} - N_{s_1} = 1, N_{s_2} - N_{t_1} = 0, N_{t_2} - N_{s_2} \geq 1) \\ = \; & e^{-\lambda s_1} \lambda(t_1 - s_1) e^{-\lambda(t_1 - s_1)} e^{-\lambda(s_2 - t_1)} \left(1 - e^{\lambda(t_2 - s_2)}\right) \\ = \; & \lambda(t_1 - s_1)\left(e^{-\lambda s_2} - e^{-\lambda t_2}\right) \\ = \; & \int_{s_1 < y_1 < t_1, s_2 < y_2 < t_2} \lambda^2 e^{-\lambda y_2} dy_1 dy_2. \end{aligned}$$

Which shows that

$$(y_1, y_2) \longmapsto \lambda^2 e^{-\lambda y_2} 1_{\{0 < y_1 < y_2\}}$$

is the density of (T_1, T_2). Since $(W_1, W_2) = (T_1, T_1 + T_2)$ it follows that the density of (W_1, W_2) is

$$\lambda^2 e^{-\lambda(w_1 + w_2)} 1_{\{w_1 > 0, W_2 > 0\}},$$

hence (4.20) by integration. ◇

Corollary 4.2 *T_n has the distribution Gamma (n, λ) with density*

$$f_n(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} 1_{\mathbb{R}_+}(t). \qquad (4.21)$$

Proof. Consider the identity

$$P(T_n > t) = P(N_t < n) = \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \; t > 0.$$

Taking derivative with respect to t we get

$$\begin{aligned} -f_n(t) &= -\lambda \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \sum_{j=1}^{n-1} e^{-\lambda t} \frac{\lambda^j t^{j-1}}{(j-1)!} \\ &= -\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \; t > 0, \end{aligned}$$

hence the desired result follows. ◇

We now show that the obtained properties in Theorem 4.2 characterize Poisson Processes.

Theorem 4.3 *Let (T_n) be a Point Process such that the random variables $W_n = T_n - T_{n-1}$, $n \geq 1$ are independent with the same exponential distribution $\mathcal{E}(\lambda)$. Then the associated Counting Process (N_t) is a Poisson Process with intensity λ.*

Proof. By hypothesis $(W_1, \cdots, W_n)$ has the density

$$\lambda^n e^{-\lambda(w_1 + \cdots + w_n)} 1_{\{w_1 > 0, \cdots, w_n > 0\}}.$$

Setting

$$t_i = w_1 + \cdots + w_i, \qquad 1 \leq i \leq n,$$

we obtain the density of $(T_1, \cdots, T_n)$:

$$f_{(T_1, \cdots, T_n)}(t_1, \cdots, t_n) = \lambda^n e^{-\lambda t_n} 1_{\{0 < t_1 < \cdots < t_n\}}. \tag{4.22}$$

Now, for convenience, we only compute the distribution of $(N_s, N_t - N_s)$, $t > s$. For that purpose we write

$$\begin{aligned} &P(N_s = k, N_t - N_s = n) = P(T_k \leq s < T_{k+n}, T_{k+n} \leq t < T_{k+n+1}) \\ &= \int_{0 < t_1 < \cdots < t_k \leq s < \cdots < t_{k+n} \leq t < t_{k+n+1}} \lambda^{k+n+1} e^{-\lambda t_{k+n+1}} dt_1 \cdots dt_{k+n+1}. \end{aligned}$$

Now it is easy to obtain the following equalities:

$$\int_t^\infty \lambda^{k+n+1} e^{-\lambda t_{k+n+1}} dt_{k+n+1} = e^{-\lambda t} \lambda^{k+n},$$

$$\int_{s < t_{k+1} < \cdots < t_{k+n} \leq t} dt_{k+1} \cdots dt_{k+n} = \frac{(t-s)^n}{n!},$$

and

$$\int_{0<t_1<\cdots<t_k\leq s} dt_1 \cdots dt_k = \frac{s^k}{k!}.$$

Combining the above results and applying Fubini's Theorem (see Appendix), we obtain

$$P(N_s = k, N_t - N_s = n) = \left(\lambda^n e^{-\lambda(t-s)} \frac{(t-s)^n}{n!}\right)$$

$$\times \quad \left(\lambda^k e^{-\lambda s} \frac{s^k}{k!}\right), \quad k = 0, 1, 2, \cdots, \quad n = 0, 1, 2, \cdots.$$

which completes the the proof of Theorem 4.3. ◇

Finally we may characterize a Poisson Process either by

$$(N_{t_1}, N_{t_1} - N_{t_1}, \cdots, N_{t_k} - N_{t_{k-1}}) \sim \mathcal{P}(\lambda t_1) \otimes \cdots \otimes \mathcal{P}(\lambda(t_k - t_{k-1}), \quad (4.23)$$

$k \geq 1$, $0 \leq t_1 < \cdots < t_k$, or by

$$(T_1, T_2 - T_1, \cdots, T_n - T_{n-1}) \sim \otimes^n \mathcal{E}(\lambda), \quad (4.24)$$

where $\sim$ means "is distributed as" and $\otimes$ denotes the product measure (see Appendix).

4.4 Some Properties of Poisson Processes

The current section is devoted to some properties which are useful for statistical studies of Poisson Processes.

a) Poisson processes and order statistics

First let us define the *order statistics* associated with i.i.d real random variables $U_1, \cdots, U_k$, as the random vector $(U_{(1)}, \cdots, U_{(k)})$ where $U_{(1)} \leq \cdots \leq U_{(k)}$ is a rearrangement of the U_i's.

The next theorem shows that $(T_1, \cdots, T_k)$ is a "conditional" order statistics.

Theorem 4.4 *Let (N_t) be a Poisson Process associated with the Point Process (T_n), then*

$$\mathcal{L}((T_1, \cdots, T_k) | N_t = k) = \mathcal{L}\left((U_{(1)}, \cdots, U_{(k)})\right), \quad k = 1, 2, \cdots; t > 0, \quad (4.25)$$

where $(U_{(1)}, \cdots, U_{(k)})$ denotes the order statistics associated with i.i.d. random variables with uniform distribution over $[0, t]$.