CHAPTER 3

Random Point Processes

3.1 BASIC CONCEPTS

A *point process* is a sequence of real numbers $\{t_1, t_2, ...\}$ with properties

$$t_1 < t_2 < \cdots \quad \text{and} \quad \lim_{i \to \infty} t_i = +\infty. \tag{3.1}$$

That means, a point process is a strictly increasing sequence of real numbers, which does not have a finite limit point. In practice, point processes occur in numerous situations: arrival time points of customers at service stations (workshops, filling stations, supermarkets, ...), failure time points of machines, time points of traffic accidents, occurrence of nature catastrophies, occurrence of supernovas,... Generally, at time point t_i a certain *event* happens. Hence, the t_i are called *event times*. With regard to the arrival of customers at service stations, the t_i are also called *arrival times*. If not stated otherwise, the assumption $t_1 \geq 0$ is made.

Although the majority of applications of point processes refer to sequences of time points, there are other interpretations as well. For instance, sequences $\{t_1, t_2, ...\}$ can be generated by the location of potholes in a road. Then t_i denotes the distance of the ith pothole from the beginning of the road. Or, the location is measured, at which an imaginary straight line, which runs through a forest stand, hits trees. (This is the base of the well-known *Bitterlich method* for estimating the total number of trees in a forest stand.) Strictly speaking, since both road and straight line through a forest stand have finite lengths, to meet assumption (3.1), they have to be considered finite samples from a point process.

A point process $\{t_1, t_2, ...\}$ can equivalently be represented by the sequence of its *interevent* (*interarrival*) times

$$\{y_1, y_2, ...\} \text{ with } y_i = t_i - t_{i-1};\ i = 1, 2, ...;\ t_0 = 0.$$

Counting Process Frequently, the event times are of less interest than the number of events, which occur in an interval $(0, t]$, $t > 0$. This number is denoted as $n(t)$:

$$n(t) = \max\{n,\ t_n \leq t\}.$$

For obvious reasons, $\{n(t), t \geq 0\}$ is said to be the *counting process* belonging to the point process $\{t_1, t_2, ...\}$. Here and in what follows, it is assumed that more than one event cannot occur at a time. Point processes with this property are called *simple.* The number of events, which occur in an interval $(s, t]$, $s < t$, is

$$n(s, t) = n(t) - n(s).$$

To be able to count the number $n(A)$ of events which occur in an arbitrary subset A of $[0, \infty)$ the indicator function of the event 't_i belongs to A' is introduced:

$$I_i(A) = \begin{cases} 1 & \text{if } t_i \in A \\ 0 & \text{otherwise} \end{cases} . \tag{3.2}$$

Then,

$$n(A) = \sum_{i=0}^{\infty} I_i(A) .$$

Example 3.1 Let be given a finite sample from a point process:

$$\{2, 4, 10, 18, 24, 31, 35, 38, 40, 44, 45, 51, 57, 59\}$$

The figures indicate the times (in seconds) at which within a time span of a minute a car passes a control point. Then, within the first 16 seconds, $n(16) = 3$ cars passed the control point, and in the interval $(31, 49]$ exactly $n(31, 49) = n(49) - n(30) = 5$ cars passed the control point. In terms of the indicator function (3.2), given the time span $A = (10, 20] \cup [51, 60]$

$$I_{18}(A) = I_{24}(A) = I_{51}(A) = I_{57}(A) = I_{59}(A) = 1,$$

$$I_i(A) = 0 \text{ for } i \neq 18,\ 24,\ 51,\ 57,\ 59.$$

Hence,

$$n(A) = \sum_{i=0}^{\infty} I_i(A) = \sum_{i=0}^{60} I_i(A) = 5 . \qquad \square$$

Recurrence Times The *forward recurrence time* of a point process $\{t_1, t_2, ...\}$ with respect to time point t is defined as

$$a(t) = t_{n+1} - t \text{ for } t_n \leq t < t_{n+1};\ n = 0, 1, ...,\ t_0 = 0. \tag{3.3}$$

Hence, $a(t)$ is the time span from t (usually interpreted as the 'presence') to the occurrence of the next event. A simpler way of characterizing $a(t)$ is

$$a(t) = t_{n(t)+1} - t . \tag{3.4}$$

$t_{n(t)}$ is the largest event time before t and $t_{n(t)+1}$ is the smallest event time after t.

The *backward recurrence time* $b(t)$ with respect to time point t is

$$b(t) = t - t_{n(t)} . \tag{3.5}$$

Thus, $b(t)$ is the time which has elapsed from the last event time before t to time t.

Marked Point Processes Frequently, in addition to their arrival times, events come with another piece of information. For instance: If t_i is the time point the ith customer arrives at a supermarket, then the customer will spend there a certain amount of money m_i. If t_i is the failure time point of a machine, then the time (or cost) m_i necessary for removing the failure may be assigned to t_i. If t_i denotes the time of the ith bank robbery in a town, then the amount m_i the robbers got away with is of interest. If t_i is the arrival time of the ith claim at an insurance company, then the size

m_i of this claim is of particular importance to the company. If t_i is the time of the ith supernova in a century, then its light intensity m_i is of interest to astronomers, and so on. This leads to the concept of a marked point process: Given a point process $\{t_1, t_2, ...\}$, a sequence of two-dimensional vectors

$$\{(t_1, m_1), (t_2, m_2), ...\} \tag{3.6}$$

with m_i being an element of a *mark space* **M** is called a *marked point process*. In most applications, as in the four examples above, the mark space **M** is a subset of the real axis $(-\infty, +\infty)$ with the respective unites of measurements attached.

Random Point Processes Usually the event times are random variables. A sequence of random variables $\{T_1, T_2, ...\}$ with

$$T_1 < T_2 < \cdots \text{ and } P(\lim_{i\to\infty} T_i = +\infty) = 1 \tag{3.7}$$

is a *random point process*. By introducing the *random interevent* (*interarrival*) *times*

$$Y_i = T_i - T_{i-1};\ i = 1, 2, ...;\ T_0 = 0,$$

a random point process can equivalently be defined as a sequence of positive random variables $\{Y_1, Y_2, ...\}$ with property

$$P(\lim_{n\to\infty} \Sigma_{i=0}^n Y_i = \infty) = 1.$$

In either case, with the terminology introduced in section 2.1, a random point process is a discrete-time stochastic process with state space $\mathbf{Z} = [0, +\infty)$. Thus, a point process (3.1) is a *sample path*, a *realization* or a *trajectory* of a random point process. A point process is called *simple* if at any time point t not more than one event can occur.

Recurrent Point Processes A random point process $\{T_1, T_2, ...\}$ is said to be *recurrent* if its corresponding sequence of interarrival times $\{Y_1, Y_2, ...\}$ is a sequence of independent, identically distributed random variables. The most important recurrent point processes are homogenous Poisson processess and renewal processes (sections 3.2.1 and 3.3).

Random Counting Processes Let

$$N(t) = \max\{n,\ T_n \le t\}$$

be the random number of events occurring in the interval $(0, t]$. Then the continuous-time stochastic process $\{N(t),\ t \ge 0\}$ with state space $\mathbf{Z} = \{0, 1, ...\}$ is called the *random counting process* belonging to the random point process $\{T_1, T_2, ...\}$. Any counting process $\{N(t),\ t \ge 0\}$ has properties

1) $N(0) = 0$,

2) $N(s) \le N(t)$ for $s \le t$,

3) For any s, t with $0 \le s < t$, the increment $N(s, t) = N(t) - N(s)$ is equal to the number of events which occur in $(s, t]$.

Conversely, every stochastic process $\{N(t),\ t \ge 0\}$ in continuous time having these three properties is the counting process of a certain point process $\{T_1, T_2, ...\}$. Thus, from the statistical point of view, the stochastic processes

$$\{T_1, T_2, ...\},\ \{Y_1, Y_2, ...\},\ \text{and}\ \{N(t),\ t \ge 0\}$$

are equivalent. For that reason, a random point process is frequently defined as a continuous-time stochastic process $\{N(t),\ t \ge 0\}$ with properties 1 to 3. Note that

$$N(t) = N(0, t).$$

The most important characteristic of a counting process $\{N(t),\ t \ge 0\}$ is the probability distribution of its increments $N(s,t) = N(t) - N(s)$, which determines for all intervals $[s, t),\ s < t,$ the probabilities

$$p_k(s,t) = P(N(s,t) = k); \quad k = 0, 1, ...$$

The mean numbers of events in $(s, t]$ is

$$m(s,t) = m(t) - m(s) = E(N(s,t)) = \sum_{k=0}^{\infty} k\, p_k(s,t). \tag{3.8}$$

With

$$p_k(t) = p_k(0,t),$$

the trend function of the counting process $\{N(t), t \ge 0\}$ is

$$m(t) = E(N(t)) = \sum_{k=0}^{\infty} k\, p_k(t), \quad t \ge 0. \tag{3.9}$$

A random counting process is called *simple* if the underlying point process is simple. Figure 3.1 shows a possible sample path of a simple random counting process.

Note In what follows the attribute 'random' is usually omitted if it is obvious from the notation or the context that random point processes or random counting processes are being dealt with.

Definition 3.1 (*stationarity*) A point process $\{T_1, T_2, ...\}$ is called *stationary* if its sequence of interarrival times $\{Y_1, Y_2, ...\}$ is strongly stationary (section 2.3), that is if for any sequence of integers $i_1, i_2, ..., i_k$ with $1 \le i_1 < i_2 < \cdots < i_k,\ k = 1, 2, ...$ and for any $\tau = 0, 1, 2, ...,$ the joint distribution functions of the following two random vectors coincide:

$$\{Y_{i_1}, Y_{i_2}, ..., Y_{i_k}\} \ \text{and}\ \{Y_{i_1+\tau}, Y_{i_2+\tau}, ..., Y_{i_k+\tau}\}. \qquad \bullet$$

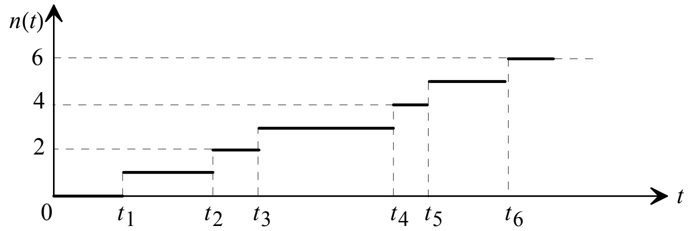

Figure 3.1 Sample path of a simple counting process

It is an easy exercise to show that if the sequence $\{Y_1, Y_2, ...\}$ is strongly stationary, the corresponding counting process $\{N(t),\ t \ge 0\}$ has homogeneous increments and vice versa. This implies the following corollary from definition 3.1:

Corollary A point process $\{T_1, T_2, ...\}$ is *stationary* if and only if its corresponding counting process $\{N(t),\ t \ge 0\}$ has homogeneous increments.

Hence, for a stationary point process, the probability distribution of any increment $N(s, t)$ depends only on the difference $\tau = t - s$:

$$p_k(\tau) = P(N(s, s+\tau) = k); \quad k = 0, 1, ...;\ s \ge 0,\ \tau > 0. \tag{3.10}$$

Thus, for a stationary point process,

$$m(\tau) = m(s, s+\tau) = m(s+\tau) - m(s) \quad \text{for all } s \ge 0,\ \tau \ge 0. \tag{3.11}$$

For having nondecreasing sample paths, neither the point process $\{T_1, T_2, ...\}$ nor its corresponding counting process $\{N(t),\ t \ge 0\}$ can be stationary as defined in section 2.3. In particular, since only simple point processes are considered, the sample paths of $\{N(t),\ t \ge 0\}$ are step functions with jump heights being equal to 1.

Remark Sometimes it is more convenient or even necessary to define random point processes as doubly infinite sequences

$$\{...,\ T_{-2}, T_1, T_0, T_1, T_2,\ ...\},$$

which tend to infinity to the left and to the right with probability 1. Then their sample paths are also doubly infinite sequences: $\{...,\ t_{-2}, t_1, t_0, t_1, t_2,\ ...\}$ and only the increments of the corresponding counting process over finite intervals are finite.

Intensity of Random Point Processes For stationary point processes, the mean number of events occurring in $[0, 1]$ is called the *intensity* of the process and will be denoted as λ. By making use of notation (3.9),

$$\lambda = m(1) = \Sigma_{k=0}^{\infty}\, k p_k(1). \tag{3.12}$$

In view of the stationarity, λ is equal to the mean number of events occurring in any interval of length 1:

$$\lambda = m(s, s+1),\ \ s \ge 0.$$

Hence, the mean number of events occurring in any interval $(s, t]$ of length $\tau = t - s$ is

$$m(s, t) = \lambda(t - s) = \lambda\tau.$$

Given a sample path $\{t_1, t_2, ...\}$ of a stationary random point process, λ is estimated by the number of events occurring in $[0, t]$ divided by the length of this interval:

$$\hat{\lambda} = n(t)/t,$$

In example 3.1, an estimate of the intensity of the underlying point process (assumed to be stationary) is $\hat{\lambda} = 14/60 \approx 0.233$.

In case of a nonstationary point process, the role of the constant intensity λ is taken over by an *intensity function* $\lambda(t)$. This function allows to determine the mean number of events $m(s,t)$ occurring in an interval $(s,t]$: For any s,t with $0 \le s < t$,

$$m(s,t)) = \int_s^t \lambda(x)\,dx\,.$$

Specifically, the mean number of events in $[0,t]$ is the trend function of the corresponding counting process:

$$m(t) = m(0,t) = \int_0^t \lambda(x)\,dx\,, \quad t \ge 0. \tag{3.13}$$

Hence, for $\Delta t \to 0$,

$$\Delta m(t) = \lambda(t)\,\Delta t + o(\Delta t)\,, \tag{3.14}$$

so that for small Δt the product $\lambda(t)\,\Delta t$ is approximately the mean number of events in $(t, t+\Delta t]$. Another interpretation of (3.14) is: If Δt is sufficiently small, then $\lambda(t)\,\Delta t$ is approximately the probability of the occurrence of an event in the interval $[t, t+\Delta t]$. Hence, the intensity function $\lambda(t)$ is the *arrival rate* of events at time t. (For *Landau's order symbol* $o(x)$, see (1.41).)

Random Marked Point Processes Let $\{T_1, T_2, ...\}$ be a random point process with random marks M_i assigned to the event times T_i. Then the sequence

$$\{(T_1, M_1),\,(T_2, M_2), ...\} \tag{3.15}$$

is called a *random marked point process*. Its (2-dimensional) sample paths are given by (3.6). The pulse process $\{(T_n, A_n);\, n = 1, 2, ...\}$ considered in example 2.5 is a special marked point processes.

Random marked point processes are dealt with in full generality in Matthes, Kerstan, and Mecke [60]. For other mathematically prestigious treatments, see, for instance, König and Schmidt [51] or Stigman [78].

Compound Stochastic Processes Let $\{(T_1, M_1),\,(T_2, M_2), ...\}$ be a random marked point process and $\{N(t),\, t \ge 0\}$ be the counting process belonging to the point process $\{T_1, T_2, ...\}$. The stochastic process $\{C(t),\, t \ge 0\}$ defined by

$$C(t) = \begin{cases} 0 & \text{for } \ 0 \le t < T_1 \\ \sum_{i=1}^{N(t)} M_i & \text{for } \qquad t \ge T_1 \end{cases}$$

is called a *compound* (*cumulative, aggregate*) *stochastic process*. According to the underlying point process, there are, for instance, compound Poisson processes and compound renewal processes. If $\{T_1, T_2, ...\}$ is a claim arrival process and M_i the size of the ith claim, then $C(t)$ is the total claim amount in $[0,t)$. If T_i is the time of the ith breakdown of a machine and M_i the corresponding repair cost, then $C(t)$ is the total repair cost in $[0,t)$.

3.2 POISSON PROCESSES

3.2.1 Homogeneous Poisson Processes

3.2.1.1 Definition and Properties

In the theory of stochastic processes, and maybe even more in its applications, the homogeneous Poisson process is just as popular as the exponential distribution in probability theory. Moreover, there is a close relationship between the homogeneous Poisson process and the exponential distribution (theorem 3.2).

Definition 3.2 (*homogeneous Poisson process*) A counting process $\{N(t),\ t \ge 0\}$ is a *homogeneous Poisson process with intensity* $\lambda,\ \lambda > 0,$ if it has the following properties:

1) $N(0) = 0,$

2) $\{N(t),\ t \ge 0\}$ is a stochastic process with independent increments.

3) Its increments $N(s,t) = N(t) - N(s),\ 0 \le s < t,$ have a Poisson distribution with parameter $\lambda(t-s)$:

$$P(N(s,t) = i) = \frac{(\lambda(t-s))^i}{i!}\, e^{-\lambda(t-s)}; \quad i = 0, 1, ..., \tag{3.16}$$

or, equivalently, introducing the length $\tau = t - s$ of the interval $[s, t]$, for all $\tau > 0$,

$$P(N(s, s+\tau) = i) = \frac{(\lambda\tau)^i}{i!}\, e^{-\lambda\tau}; \quad i = 0, 1, ... \tag{3.17}$$

●

(3.16) implies that the homogeneous Poisson process has homogeneous increments. Thus, the corresponding *Poisson point process* $\{T_1, T_2, ...\}$ is stationary in the sense of definition 3.1.

Theorem 3.1 A counting process $\{N(t),\ t \ge 0\}$ with $N(0) = 0$ is a homogeneous Poisson process with intensity λ if and only if it has the following properties:

a) $\{N(t),\ t \ge 0\}$ has homogeneous and independent increments.

b) The process is *simple,* i.e. $P(N(t, t+h) \ge 2) = o(h)$.

c) $P(N(t, t+h) = 1) = \lambda\, h + o(h)$.

Proof To prove that definition 3.2 implies properties a), b) and c), it is only necessary to show that a homogeneous Poisson process satisfies properties b) and c).

The simplicity of the Poisson process easily results from (3.17):

$$P(N(t, t+h) \ge 2) = e^{-\lambda h} \sum_{i=2}^{\infty} \frac{(\lambda h)^i}{i!}$$

$$= \lambda^2 h^2 e^{-\lambda h} \sum_{i=0}^{\infty} \frac{(\lambda h)^i}{(i+2)!} \le \lambda^2 h^2 = o(h).$$

Another application of (3.17) and the simplicity of the Poisson process proves c):

$$\begin{aligned} P(N(t,t+h)=1) &= 1-P(N(t,t+h)=0)-P(N(t,t+h)\ge 2) \\ &= 1-e^{-\lambda h}+o(h) = 1-(1-\lambda h)+o(h) \\ &= \lambda h + o(h). \end{aligned}$$

Conversely, it needs to be shown that a stochastic process with properties a), b) and c) is a homogeneous Poisson process. In view of the assumed homogeneity of the increments, it is sufficient to prove the validity of (3.17) for $s=0$. Thus, letting

$$p_i(t) = P(N(0,t)=i) = P(N(t)=i)\,; \quad i=0,1,\dots$$

it is to show that

$$p_i(t) = \frac{(\lambda t)^i}{i!}\, e^{-\lambda t}; \quad i=0,1,\dots. \tag{3.18}$$

From a),

$$\begin{aligned} p_0(t+h) &= P(N(t+h)=0) = P(N(t)=0,\ N(t,t+h)=0) \\ &= P(N(t)=0)\,P(\,N(t,t+h)=0) = p_0(t)\,p_0(h)\,. \end{aligned}$$

In view of b) and c), this result implies

$$p_0(t+h) = p_0(t)(1-\lambda h)+o(h)$$

or, equivalently,

$$\frac{p_0(t+h)-p_0(t)}{h} = -\lambda p_0(t)+o(h).$$

Taking the limit as $h\to 0$ yields

$$p_0'(t) = -\lambda p_0(t)\,.$$

Since $p_0(0)=1$, the solution of this differential equation is

$$p_0(t) = e^{-\lambda t}, \quad t\ge 0,$$

so that (3.18) holds for $i=0$.

Analogously, for $i\ge 1$,

$$\begin{aligned} p_i(t+h) &= P(N(t+h)=i) \\ &= P(N(t)=i,\ N(t+h)-N(t)=0) + P(N(t)=i-1,\ N(t+h)-N(t)=1) \\ &\quad + \textstyle\sum_{k=2}^{i}\ P(N(t)=k,\ N(t+h)-N(t)=i-k)\,. \end{aligned}$$

Because of c), the sum in the last row is $o(h)$. Using properties a) and b),

$$\begin{aligned} p_i(t+h) &= p_i(t)\,p_0(h)+p_{i-1}(t)\,p_1(h)+o(h) \\ &= p_i(t)\,(1-\lambda h)+p_{i-1}(t)\,\lambda h+o(h), \end{aligned}$$

or, equivalently,

$$\frac{p_i(t+h)-p_i(t)}{h} = -\lambda\,[p_i(t)-p_{i-1}(t)] + o(h).$$

Taking the limit as $h \to 0$ yields a system of linear differential equations in the $p_i(t)$

$$p_i'(t) = -\lambda[p_i(t)-p_{i-1}(t)]; \quad i = 1, 2, ... \tag{3.19}$$

Starting with $p_0(t) = e^{-\lambda t}$, the solution (3.18) is obtained by induction. ■

The practical importance of theorem 3.1 is that the properties a), b) and c) can be verified without any quantitative investigations, only by qualitative reasoning based on the physical or other nature of the process. In particular, the simplicity of the homogeneous Poisson process implies that the occurrence of more than one event at the same time has probability 0.

Note Throughout this chapter, those events, which are counted by a Poisson process $\{N(t), t \ge 0\}$, will be called *Poisson events*.

Let $\{T_1, T_2, ...\}$ be the point process, which belongs to the homogeneous Poisson process $\{N(t), t \ge 0\}$, i.e. T_n is the random time point at which the nth Poisson event occurs. The obvious relationship

$$T_n \le t \text{ if and only if } N(t) \ge n$$

implies

$$P(T_n \le t) = P(N(t) \ge n)\,. \tag{3.20}$$

Therefore, T_n has distribution function

$$F_{T_n}(t) = P(N(t) \ge n) = \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t}; \quad n = 1, 2, ... \tag{3.21}$$

Differentiation of $F_{T_n}(t)$ with respect to t yields the density of T_n:

$$f_{T_n}(t) = \lambda e^{-\lambda t} \sum_{i=n}^{\infty} \frac{(\lambda t)^{i-1}}{(i-1)!} - \lambda e^{-\lambda t} \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!}\,.$$

On the right-hand side of this equation, all terms but one cancel:

$$f_{T_n}(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}; \quad t \ge 0, \ n = 1, 2, ... \tag{3.22}$$

Thus, T_n has an Erlang distribution with parameters n and λ. In particular, T_1 has an exponential distribution with parameter λ and the interevent times

$$Y_i = T_i - T_{i-1}; \ i = 1, 2, ...; \ k = 1, 2, ...; \ T_0 = 0.$$

are independent and identically distributed as T_1 (see example 1.23). Moreover,

$$T_n = \sum_{i=1}^{n} Y_i.$$

These results yield the most simple and, at the same time, the most important characterization of the homogeneous Poisson process:

Theorem 3.2 Let $\{N(t), t \geq 0\}$ be a counting process and $\{Y_1, Y_2, ...\}$ be the corresponding sequence of interarrival times. Then $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with intensity λ if and only if the $Y_1, Y_2, ...$ are independent, exponentially with parameter λ distributed random variables. ■

The counting process $\{N(t), t \geq 0\}$ is statistically equivalent to both its corresponding point process $\{T_1, T_2, ...\}$ of event times and the sequence of interarrival times $\{Y_1, Y_2, ...\}$. Hence, $\{T_1, T_2, ...\}$ and $\{Y_1, Y_2, ...\}$ are sometimes also called Poisson processes.

Example 3.2 From previous observations it is known that the number of traffic accidents $N(t)$ in an area over the time interval $[0, t)$ can be described by a homogeneous Poisson process $\{N(t), t \geq 0\}$. On an average, there is one accident within 4 hours, i.e. the intensity of the process is

$$\lambda = 0.25\,[h^{-1}].$$

(1) What is the probability p of the event (time unit: hour)

"at most one accident in [0, 10), at least two accidents in [10, 16), and no accident in [16, 24)"?

This probability is

$$p = P(N(10) - N(0) \leq 1,\ N(16) - N(10) \geq 2,\ N(24) - N(16) = 0).$$

In view of the independence and the homogeneity of the increments of $\{N(t), t \geq 0\}$, p can be determined as follows:

$$p = P(N(10) - N(0) \leq 1)\, P(N(16) - N(10) \geq 2)\, P(N(24) - N(16) = 0)$$

$$= P(N(10) \leq 1)\, P(N(6) \geq 2)\, P(N(8) = 0)\,.$$

Now,

$$P(N(10) \leq 1) = P(N(10) = 0) + P(N(10) = 1)$$

$$= e^{-0.25 \cdot 10} + 0.25 \cdot 10 \cdot e^{-0.25 \cdot 10} = 0.2873,$$

$$P(N(6) \geq 2) = 1 - e^{-0.25 \cdot 6} - 0.25 \cdot 6 \cdot e^{0.25 \cdot 6} = 0.4422,$$

$$P(N(8) = 0) = e^{-0.25 \cdot 8} = 0.1353.$$

Hence, the desired probability is $p = 0.0172$.

(2) What is the probability that the 2 nd accident occurs not before 5 hours?

Since T_2, the random time of the occurrence of the second accident, has an Erlang distribution with parameters $n = 2$ and $\lambda = 0.25$,

$$P(T_2 > 5) = 1 - F_{T_2}(5) = e^{-0.25 \cdot 5}(1 + 0.25 \cdot 5).$$

Thus, $P(T_2 > 5) = 0.6446$. □

The following examples make use of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad x \in (-\infty, +\infty).$$

Example 3.3 (*random telegraph signal*) A random signal $X(t)$ have structure

$$X(t) = Y(-1)^{N(t)}, \ t \ge 0, \tag{3.23}$$

where $\{N(t), \ t \ge 0\}$ is a homogeneous Poisson process with intensity λ and Y is a binary random variable with

$$P(Y = 1) = P(Y = -1) = 1/2,$$

which is independent of $N(t)$ for all t. Signals of this structure are called *random telegraph signals.* Random telegraph signals are basic modules for generating signals with a more complicated structure. Obviously, $X(t) = 1$ or $X(t) = -1$ and Y determines the sign of $X(0)$. Figure 3.2 shows a sample path $x = x(t)$ of the stochastic process $\{X(t), \ t \ge 0\}$ on condition $Y = 1$ and $T_n = t_n; \ n = 1, 2, ...$

$\{X(t), \ t \ge 0\}$ is wide-sense stationary. To see this, firstly note that

$$|X(t)|^2 = 1 < \infty \ \text{ for all } t \ge 0.$$

Hence, $\{X(t), t \ge 0\}$ is a second-order process. With

$$I(t) = (-1)^{N(t)},$$

its trend function is $m(t) = E(X(t)) = E(Y)\,E(I(t))$. Hence, since $E(Y) = 0$,

$$m(t) \equiv 0.$$

It remains to show that the covariance function $C(s, t)$ of this process depends only on $|t - s|$. This requires knowledge of the probability distribution of $I(t)$: A transition from $I(t) = -1$ to $I(t) = +1$ or, conversely, from $I(t) = +1$ to $I(t) = -1$ occurs at those time points, at which Poisson events occur, i.e. when $N(t)$ jumps:

$$P(I(t) = 1) = P(\text{even number of jumps in } [0, t])$$

$$= e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^{2i}}{(2i)!} = e^{-\lambda t} \cosh \lambda t,$$

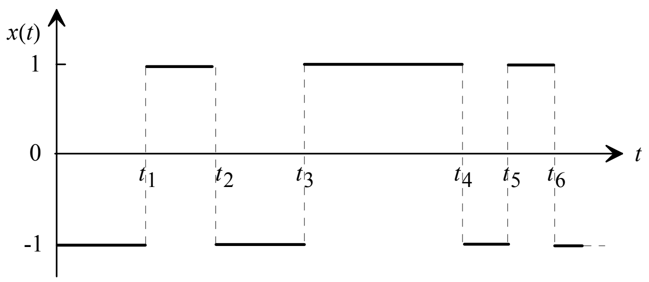

Figure 3.2 Sample path of the random telegraph signal

Analogously,

$$P(I(t) = -1) = P(\text{odd number of jumps in } [0, t])$$
$$= e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^{2i+1}}{(2i+1)!} = e^{-\lambda t} \sinh \lambda t\,.$$

Hence the mean value of $I(t)$ is

$$E[I(t)] = 1 \cdot P(I(t) = 1) + (-1) \cdot P(I(t) = -1)$$
$$= e^{-\lambda t}[\cosh \lambda t - \sinh \lambda t] = e^{-2\lambda t}.$$

Since

$$C(s, t) = Cov\,[X(s), X(t)]$$
$$= E[(X(s)\, X(t))] = E[Y I(s)\, Y I(t)]$$
$$= E[Y^2\, I(s)\, I(t)] = E(Y^2)\, E[I(s)\, I(t)]$$

and $E(Y^2) = 1$, the covariance function of $\{X(t),\ t \ge 0\}$ has structure

$$C(s, t) = E[I(s)\, I(t)]\,.$$

Thus, in order to evaluate $C(s, t)$, the joint distribution of $(I(s), I(t))$ has to be determined: From (1.6) and the homogeneity of the increments of $\{N(t),\ t \ge 0\}$, assuming $s < t$,

$$p_{1,1} = P(I(s) = 1,\, I(t) = 1) = P(I(s) = 1) P(\, I(t) = 1 \,|\, I(s) = 1)$$
$$= e^{-\lambda s} \cosh \lambda s\, P(\text{even number of jumps in } (s, t])$$
$$= e^{-\lambda s} \cosh \lambda s\, e^{-\lambda (t-s)} \cosh \lambda (t-s)$$
$$= e^{-\lambda t} \cosh \lambda s\ \cosh \lambda (t-s)\,.$$

Analogously,

$$p_{1,-1} = P(I(s) = 1, I(t) = -1) \quad = e^{-\lambda t} \cosh \lambda s\ \sinh \lambda (t-s)\,,$$
$$p_{-1,1} = P(I(s) = -1, I(t) = 1) \quad = e^{-\lambda t} \sinh \lambda s\ \sinh \lambda (t-s)\,,$$
$$p_{-1,-1} = P(I(s) = -1, I(t) = -1) = e^{-\lambda t} \sinh \lambda s\ \cosh \lambda (t-s)\,.$$

Now

$$E[I(s)I(t)] = p_{1,1} + p_{-1,-1} - p_{1,-1} - p_{-1,1}\,,$$

so that

$$C(s, t) = e^{-2\lambda (t-s)},\ \ s < t.$$

Since the order of s and t can be changed,

$$C(s, t) = e^{-2\lambda |t-s|}\,.$$

Hence, the random telegraph signal $\{X(t),\ t \ge 0\}$ is a weakly stationary process. □