# A Useful Theorem for Nonlinear Devices Having Gaussian Inputs*

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**Summary—If and only if the inputs to a set of nonlinear, zero-memory devices are variates drawn from a Gaussian random process, a useful general relationship may be found between certain input and output statistics of the set. This relationship equates partial derivatives of the (high-order) output correlation coefficient taken with respect to the input correlation coefficients, to the output correlation coefficient of a new set of nonlinear devices bearing a simple derivative relation to the original set. Application is made to the interesting special cases of conventional cross-correlation and autocorrelation functions, and Bussgang's theorem is easily proved. As examples, the output autocorrelation functions are simply obtained for a hard limiter, linear detector, clipper, and smooth limiter.**

IN THE COURSE of investigating the asymptotic frequency behavior of power spectra resulting from the passage of noise through zero-memory nonlinear devices, an interesting, unique property of Gaussian processes has been encountered, which does not appear to have been previously reported.

## Statement of the Theorem

Assume $x_1, x_2, \cdots, x_n$ to be random variables from a Gaussian process whose $n$th order joint probability density is given by:[1]

$$p(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \mid M_n \mid^{-1/2} \cdot \exp\left\{-\frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{M_{rs}}{\mid M_n \mid} (x_r - \overline{x_r})(x_s - \overline{x_s})\right\} \quad (1)$$

where $\mid M_n \mid$ is the determinant of $M_n = [\rho_{rs}]$ and $\rho_{rs} = \overline{x_r x_s} - \overline{x_r}\,\overline{x_s} = \rho_{sr}$ is the correlation coefficient of $x_r$ and $x_s$. The means of $x_r$ and $x_s$ are $\overline{x_r}$ and $\overline{x_s}$, respectively. $M_{rs}$ is the cofactor of $\rho_{sr}$ in $M_n$.

Let there be $n$ zero-memory nonlinear devices (linearity of course being included as a special case) specified by the input-output relationship $f_i(x)$, $i = 1, 2, \cdots, n$. Let each $x_i$ be the single input to a corresponding $f_i(x)$, and designate the $n$th-order correlation coefficient of the outputs as:

$$R = \overline{\prod_{i=1}^{n} f_i(x_i)} \quad (2)$$

where the bar denotes the average taken over all $x_i$. Then, with weak restrictions on the $f_i(x)$, we have the following theorem for the partial derivatives of $R$ with respect to the input correlation coefficients:

$$\frac{\partial^k R}{\prod_{m=1}^{N} (\partial \rho_{r_m s_m})^{k_m}} = \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \overline{\left[\prod_{i=1}^{n} f_i^{\left(\sum_{m=1}^{N} \epsilon_{im} k_m\right)}(x_i)\right]} \quad (3)$$

where $r_m$ and $s_m$, $m = 1, 2, \cdots, N$, are integers lying between 1 and $n$, inclusive, and are not necessarily distinct. The $k_m$ are positive integers, with $k = \sum_{m=1}^{N} k_m$. $\epsilon_{im}$ is the number of times $i$ appears in $(r_m, s_m)$. $\delta_{r_m s_m}$ is the Kronecker $\delta$ function, $\delta_{r_m s_m} = 1$ for $r_m = s_m$, 0 for $r_m \neq s_m$. The symbol $f_i^{(q)}(x_i)$ denotes the $q$th derivative of $f_i(x)$, taken at $x_i$.

Furthermore, not only is the above theorem true for inputs having an $n$th-order joint Gaussian distribution, but it holds true *only* for such inputs if the $f_i(x)$ are allowed to be of general form.

### *Proof*

We now prove that in order for (3) to be satisfied it is both sufficient and necessary that the $x_i$ have the joint probability density given by (1). Assume that each $f_i(x)$ can be represented by the sum of two Laplace transforms,[2]

$$f_i(x) = \frac{1}{2\pi j} \int_{C_{i+}} h_{i+}(u) e^{jux}\, du + \frac{1}{2\pi j} \int_{C_{i-}} h_{i-}(u) e^{jux}\, du \quad (4)$$

where

$$\left.\begin{aligned} h_{i+}(u) &= \int_0^{\infty} f_i(x) e^{-jux}\, dx \\ h_{i-}(u) &= \int_{-\infty}^{0} f_i(x) e^{-jux}\, dx \end{aligned}\right\} \quad (5)$$

and the $C_{i+}$ and $C_{i-}$ are appropriate contours. Without assuming any particular form for $p(x_1, x_2, \cdots, x_n)$ for the present,

$$R = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdot \prod_{i=1}^{n} f_i(x_i) p(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n. \quad (6)$$

Substituting (4) in (6) and inverting the order of integration, following Rice's characteristic function method,[3]

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[1] H. Cramér, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., sec. 24.2; 1946.

[2] D. V. Widder, "The Laplace Transform," Princeton University Press, Princeton, N. J., ch. 6; 1946.

[3] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, July, 1944; and vol. 24, pp. 46–156; January, 1945. See sec. 4.8.

$$R = \frac{1}{(2\pi j)^n} \sum{}' \int_{C_{1*}} \int_{C_{2*}} \cdots \int_{C_{n*}} \cdot \prod_{i=1}^{n} h_{i*}(u_i)\theta(u_1, u_2, \cdots, u_n)\, du_1\, du_2 \cdots du_n \qquad (7)$$

where $\sum'$ denotes a summation over all possible $\pm$ combinations and $\theta(u_1, u_2, \cdots, u_n)$ is the $n$th-order characteristic function:

$$\theta(u_1, u_2, \cdots, u_n) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(x_1, x_2, \cdots, x_n) \cdot \exp\left(j \sum_{i=1}^{n} u_i x_i\right) dx_1\, dx_2 \cdots dx_n \qquad (8)$$

with $j = \sqrt{-1}$.

We find a necessary condition for (3) to be satisfied by setting $N = 1 = k = k_1$. The partial derivative of the left-hand side of (3) is taken on $\theta$ in the integrand of (7), and the derivatives of the right-hand side are taken using (4). Thus the necessary condition:

$$\sum{}' \int_{C_{1*}} \int_{C_{2*}} \cdots \int_{C_{n*}} \prod_{i=1}^{n} h_{i*}(u_i)\left\{\frac{\partial \theta(u_1, u_2, \cdots, u_n)}{\partial \rho_{r_1 s_1}} + \left(\frac{1}{2}\right)^{\delta_{r_1 s_1}} u_{r_1} u_{s_1} \theta(u_1, u_2, \cdots, u_n)\right\} du_1\, du_2 \cdots du_n = 0 \qquad (9)$$

is obtained. The term in braces must be zero in order to satisfy (9) for arbitrary $f_i(x)$ and hence $h_{i*}(u)$. Integrating the resulting equation for all $(r_1, s_1)$ (but taking into account that $\rho_{rs} = \rho_{sr}$),

$$\log \theta(u_1, u_2, \cdots, u_n) = -\frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \rho_{rs} u_r u_s + g(u_1, u_2, \cdots, u_n) \qquad (10)$$

where $g$ is some function which must now be found.

Let $\rho_{rs} = 1$ for all $(r, s)$. Then all the $x_i$ are completely correlated, and $p(x_1, x_2, \cdots, x_n)$ can be written:

$$p(x_1, x_2, \cdots, x_n) = p(x_1) \prod_{i=2}^{n} \delta(x_i - x_1 + \overline{x_1} - \overline{x_i}) \qquad (11)$$

where $\delta(x)$ is the Dirac $\delta$ function. Substituting (11) in (8), $\theta$ is of the form:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left(j \sum_{i=1}^{n} u_i \overline{x_i}\right) g_1\left(\sum_{i=1}^{n} u_i\right), \quad \text{for all } \rho_{rs} = 1 \qquad (12)$$

where

$$g_1(u) = \int_{-\infty}^{+\infty} p_1(x_1 - \overline{x_1}) e^{ju(x_1 - \overline{x_1})}\, d(x_1 - \overline{x_1}). \qquad (13)$$

Similarly, when $\rho_{11} = 1$ $r$, $\rho_{1r} = \rho_{r1} = -1$ for all $r \neq 1$, and $\rho_{rs} = 1$ for all $r$ or $s \neq 1$, then $x_2, x_3, \cdots, x_n$ are completely correlated with $(-x_1)$ and we obtain:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left(j \sum_{i=1}^{n} u_i \overline{x_i}\right) g_1\left(u_1 - \sum_{i=2}^{n} u_i\right),$$
$$\text{for } \rho_{11} = 1,\ \rho_{1r} = \rho_{r1} = -1 \text{ for all } r \neq 1, \text{ and } \rho_{rs} = 1 \text{ for all } r, \quad s \neq 1. \qquad (14)$$

Substituting (12) in (10), we find:

$$g(u_1, u_2, \cdots, u_n) = j \sum_{i=1}^{n} u_i \overline{x_i} + g_2\left(\sum_{i=1}^{n} u_i\right) \qquad (15)$$

where $g_2(u) = \log g_1(u) + u^2/2$. On the other hand, substituting (14) in (10) yields

$$g(u_1, u_2, \cdots, u_n) = j \sum_{i=1}^{n} u_i \overline{x_i} + g_2\left(2u_1 - \sum_{i=1}^{n} u_i\right). \qquad (16)$$

Since $u_1$ and $\sum_{i=1}^{n} u_i$ may be considered as independent variables, the only solution which renders (15) and (16) compatible is $g_2(u) = K$, a constant. Thus, finally, we have from (10) and (15) the necessary condition:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left[-\frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \rho_{rs} u_r u_s + j \sum_{i=1}^{n} u_i \overline{x_i} + K\right]. \qquad (17)$$

This is recognized to be the characteristic function of the $n$-dimensional Gaussian distribution[4] of (1) ($K = 0$ for proper normalization).

It is now a simple matter to prove the sufficiency of (17), and hence (1), for satisfying (3). Using (17) in (7), and remembering that $\rho_{rs} = \rho_{sr}$,

$$\frac{(-1)^k\, \partial k_R}{\prod_{m=1}^{N} (\partial \rho_{r_m s_m})^{k_m}} = \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \sum{}' \int_{C_{1*}} \int_{C_{2*}} \cdots \int_{C_{n*}} \cdot \prod_{i=1}^{n} u_1^{\sum_{m=1}^{N} \epsilon_{imk_m}} h_{i*}(u_i)\theta(u_1,u_2,u_n)\, du_1\, du_2 \cdots du_n. \qquad (18)$$

By analogy to (6) and (7), and differentiating (4) with respect to $x$, the right side of (18) is seen to be equal to

$$(-1)^k \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdot \prod_{i=1}^{n} f_i^{\left(\sum_{m=1}^{N} \epsilon_{imk_m}\right)}(x_i) p(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n \qquad (19)$$

thus yielding (3).

## A Special Case and Its Applications

Consider the familiar situation where $n = 2$, and let $\rho$ denote the crosscorrelation coefficient of $x_1$ and $x_2$. Then (3) yields

$$\frac{\partial^k R}{\partial \rho^k} = \overline{f_1^{(k)}(x_1) f_2^{(k)}(x_2)}. \qquad (20)$$

Suppose that $x_1$ and $x_2$ are values of a stationary Gaussian time series $x(t)$ whose autocorrelation function is $\rho(\tau)$. $x_1$ is taken at time $t$ and $x_2$ at time $(t + \tau)$. $R(\tau)$ will denote the crosscorrelation function between the outputs

[4] Cramér, *op. cit.*, sec. 24.1.

of two zero-memory nonlinear devices whose inputs are $x(t)$ and $x(t+\tau)$, respectively.

Taking the particular case where $\overline{x(t)} = 0$, $\overline{x^2(t)} = 1$, and using (1),[5]

$$\frac{\partial^k R(\tau)}{\partial \rho(\tau)^k} = \overline{f_1^{(k)}[x(t)]f_2^{(k)}[x(t+\tau)]} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f_1^{(k)}(x_1)f_2^{(k)}(x_2)\exp\{-[x_1^2+x_2^2-2\rho(\tau)x_1x_2]/2[1-\rho^2(\tau)]\}}{2\pi\sqrt{1-\rho^2(\tau)}}\,dx_1\,dx_2. \tag{21}$$

Eq. (21) is particularly simple when the $f_i(x)$ are piecewise-polynomial functions and $k$ is sufficiently high. Then the $f_i^{(k)}(x)$ consist entirely of $\delta$ functions of various orders and the integral can be easily evaluated.

It is often of interest to obtain the derivatives of a crosscorrelation function with respect to $\tau$. It is convenient to break down such $\tau$ derivatives into a series of products of derivatives of $R(\tau)$ with respect to $\rho(\tau)$, and $\rho(\tau)$ with respect to $\tau$, using

$$\frac{dR(\tau)}{d\tau} = \frac{\partial R(\tau)}{\partial\rho(\tau)}\cdot\frac{d\rho(\tau)}{d\tau}. \tag{22}$$

This enables the nonlinear devices to be treated independently of the shape of the input correlation function $\rho(\tau)$, using (21). Similarly, the derivatives of $\rho(\tau)$ with respect to $\tau$ do not involve the $f_i(x)$.

As an example, Cohen[6] shows that in general, for autocorrelation functions $R(\tau)$, the limiting behavior of the corresponding power spectrum $\Phi(\omega)$ is given by:

$$\lim_{\omega\to\infty}\omega^2\Phi(\omega) = -\frac{1}{\pi}\frac{dR(\tau)}{d\tau}\bigg|_{\tau=0+}$$
$$\lim_{\omega\to\infty}\omega^2\left[\omega^2\Phi(\omega)+\frac{1}{\pi}\frac{dR(\tau)}{d\tau}\bigg|\right]_{\tau=0+} = -\frac{1}{\pi}\frac{d^3R(\tau)}{d\tau^3}\bigg|_{\tau=0+} \tag{23}$$

and so on, where the derivatives are with respect to $\tau$.

Another application of (20) is in deriving Bussgang's interesting result[7] that the crosscorrelation function between the input and the output of a nonlinear device driven by Gaussian noise has the same shape as the input autocorrelation function. In this case $f_1(x) = x$ and $f_2(x)$ is arbitrary. Then $f_1'(x)$ is unity, and all higher derivatives of $f_1(x)$ are zero. Putting this into (21) and evaluating the integral,

$$\frac{\partial^k R(\tau)}{\partial\rho(\tau)^k} = \begin{cases}\int_{-\infty}^{+\infty}\frac{f_2'(x)\exp(-x^2/2)}{\sqrt{2\pi}}\,dx & k=1\\ 0 & k>1.\end{cases} \tag{24}$$

We find easily

$$R(\tau) = \rho(\tau)\int_{-\infty}^{+\infty}\frac{xf_2(x)e^{-x^2/2}}{\sqrt{2\pi}}\,dx \tag{25}$$

thus yielding Bussgang's result. Unlike Bussgang's theorem, (20) cannot be generalized to hold for probability distributions other than Gaussian.[8–10]

## Some Simple Autocorrelation Examples [for $\overline{x(t)} = 0$, $\overline{x^2(t)} = 1$]

### *Hard Limiter*

Van Vleck's well-known result on the autocorrelation function of the output of a hard limiter[11] can be derived very simply, using (21). If

$$f_1(x) = f_2(x) = \begin{cases}1; & x\ge 0\\ -1; & x<0\end{cases} \tag{26}$$

then $f_1^{(1)}(x)$ and $f_2^{(1)}(x)$ are first-order $\delta$ functions of area 2, at $x = 0$.

Substituting in (21) and integrating,

$$\frac{\partial R(\tau)}{\partial\rho(\tau)} = \frac{2}{\pi\sqrt{1-\rho^2(\tau)}}. \tag{27}$$

When $\rho(\tau) = 0$, $R(\tau) = 0$. Thus

$$R(\tau) = \frac{2}{\pi}\int_0^{\rho(\tau)}\frac{d\rho(\tau)}{\sqrt{1-\rho^2(\tau)}} = \frac{2}{\pi}\sin^{-1}[\rho(\tau)] \tag{28}$$

which is Van Vleck's result.

### *Linear Detector*

Similarly, the autocorrelation function of the output of a linear detector can be easily found. If

$$f_1(x) = f_2(x) = \begin{cases}x; & x\ge 0\\ 0; & x<0\end{cases} \tag{29}$$

then $f_1^{(2)}(x)$ and $f_2^{(2)}(x)$ are first-order $\delta$ functions of area unity at $x = 0$. Substituting in (21) and integrating:

$$\frac{\partial^2 R(\tau)}{\partial\rho(\tau)^2} = \frac{1}{2\pi\sqrt{1-\rho^2(\tau)}}. \tag{30}$$

Doubly-integrating (30) with the boundary conditions:

[5] I. N. Amiantov and V. I. Tikhonov, "The effect of normal fluctuations on typical nonlinear elements," *Bull. Acad. Sci. USSR*, pp. 33–42; April, 1956. Here, an autocorrelation case $[f_1(x) = f_2(x)]$ is studied by expanding the second-order joint Gaussian probability density of $x(t)$ and $x(t+\tau)$ in powers of $\rho(\tau)$, using Mehler's formula; see Bateman Manuscript Project, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2, p. 194, (22); 1953. They then integrate by parts to obtain the output correlation function in a similar series but do not recognize the simple form (21) for this series. Using this method it is not required that $f_1(x)$ be Laplace transformable, rather than our proof.

[6] R. Cohen, "Some Analytical and Practical Aspects of Wiener's Theory of Prediction," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rep. No. 69, ch. 4, sec. 2; June 2, 1948.

[7] J. J. Bussgang, "Crosscorrelation Functions of Amplitude-Distorted Gaussian Signals," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rep. 216, sec. 3; March 26, 1952.

[8] J. F. Barrett and D. G. Lampard, "An expansion for some second-order probability distributions and its application to noise problems," IRE Trans. on Information Theory, vol. IT-1, pp. 10–15; March, 1955.

[9] J. L. Brown, Jr., "On a cross-correlation property for stationary random processes," IRE Trans. on Information Theory, vol. IT-3, pp. 28–31; March, 1957.

[10] A. H. Nuttall, "Invariance of Correlation Functions under Nonlinear Transformations," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Quart. Progress Rep., p. 63; October 15, 1957.

[11] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N. Y., p. 58; 1950.

$$\left.\begin{aligned}\frac{\partial R(\tau)}{\partial \rho(\tau)} &= \left[\int_0^\infty f_1^{(1)}(x)\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right]^2 = \frac{1}{4}\\ R(\tau) &= \left[\int_0^\infty \frac{xe^{-x^2/2}}{\sqrt{2\pi}}\,dx\right]^2 = \frac{1}{2\pi}\end{aligned}\right\}\ \text{for } \rho(\tau) = 0 \quad (31)$$

we obtain:

$$R(\tau) = \int_0^{\rho(\tau)} \left[\frac{1}{4} + \int_0^y \frac{dt}{2\pi\sqrt{1-t^2}}\right] dy + \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\left\{\rho(\tau)\cos^{-1}[\rho(\tau)] + \sqrt{1-\rho^2(\tau)}\right\} \quad (32)$$

which is in agreement with Rice's result.[12]

*Clipper*

The relations derived independently by Robin[13] and Laning and Battin[14] for the autocorrelation function of the output of a clipper may also be found by this method. With a clipper characteristic:

$$f_1(x) = f_2(x) = \begin{cases} l; & l \le x \\ x; & -l \le x \le l \\ -l; & x \le l \end{cases} \quad (33)$$

and $f_1^{(2)}(x)$ and $f_2^{(2)}(x)$ each are a pair of first-order $\delta$ functions at $x = -l$ and $x = l$, with areas 1 and $-1$, respectively. Substituting in (21) and integrating,

$$\frac{\partial^2 R(\tau)}{\partial \rho(\tau)^2} = \frac{\exp\left[-\frac{l^2}{1+\rho(\tau)}\right] - \exp\left[-\frac{l^2}{1-\rho(\tau)}\right]}{\pi\sqrt{1-\rho^2(\tau)}} \quad (34)$$

which is Robin's result, for input noise of unit variance.

*Smooth Limiter*

Finally, Baum's recent interesting result[15] for the autocorrelation function of the output of a device having an error-function characteristic will be derived. With

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2l^2}\,dt \quad (35)$$

we have

$$f_1^{(1)}(x) = f_2^{(1)}(x) = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2l^2}. \quad (36)$$

Substituting in (21):

$$\frac{\partial R(\tau)}{\partial \rho(\tau)} = \frac{1}{2\pi}\sqrt{\frac{\rho_1^2 - \rho_2^2}{1-\rho^2(\tau)}}$$

$$\cdot\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp\left[-\frac{\rho_1 x_1^2 + \rho_1 x_2^2 - 2\rho_2 x_1 x_2}{2(\rho_1^2-\rho_2^2)}\right]}{2\pi\sqrt{\rho_1^2-\rho_2^2}}\,dx_1\,dx_2\right\} \quad (37)$$

where

$$\left.\begin{aligned}\rho_1 &= \frac{\{l^{-2}[1-\rho^2(\tau)]+1\}[1-\rho^2(\tau)]}{\{l^{-2}[1-\rho^2(\tau)]+1\}^2-\rho^2(\tau)}\\ \rho_2 &= \frac{\rho(\tau)[1-\rho^2(\tau)]}{\{l^{-2}[1-\rho^2(\tau)]+1\}^2-\rho^2(\tau)}\end{aligned}\right\} \quad (38)$$

The term in braces in (37) must equal unity, since it is the integral of a second-order Gaussian probability density. Thus, from (38),

$$\frac{\partial R(\tau)}{\partial \rho(\tau)} = \frac{1}{2\pi}\sqrt{\frac{\rho_1^2-\rho_2^2}{1-\rho^2(\tau)}}$$

$$= \frac{1}{2\pi\sqrt{(1+l^{-2})^2 - l^{-4}\rho^2(\tau)}}. \quad (39)$$

Integrating and using the condition that when $\rho(\tau) = 0$, $R(\tau) = 0$,

$$R(\tau) = \frac{l^2}{2\pi}\int_0^{\rho(\tau)} \frac{d\rho(\tau)}{\sqrt{(l^2+1)^2-\rho^2(\tau)}}$$

$$= \frac{l^2}{2\pi}\sin^{-1}\left[\frac{\rho(\tau)}{1+l^2}\right] \quad (40)$$

which is in agreement with Baum's result.

[12] Rice, *op. cit.*, eq. (4.7–5).

[13] L. Robin, "The autocorrelation function and power spectrum of clipped thermal noise. Filtering of simple periodic signals in this noise," *Ann. Telecomm.*, vol. 7, pp. 375–387; September, 1952.

[14] J. H. Laning, Jr. and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., p. 362, eq. (B-8); 1956.

[15] R. F. Baum, "The correlation function of smoothly limited Gaussian noise," IRE Trans. on Information Theory, vol. IT-3, pp. 193–197; September, 1957.