A Modified Version of Price's Theorem

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Abstract—Price's theorem for zero-memory nonlinear devices with Gaussian inputs is modified to make the theorem more amenable to the analysis of functions of more than two Gaussian variates. In the general case a *single* ordinary differential equation for the output correlation function is derived. Several examples illustrate the applicability of the modified version.

I. Introduction

In 1958 Price[1] stated and proved a theorem yielding either ordinary or partial differential equations for the expected value of a nonlinear function of jointly distributed Gaussian random variables. This theorem was later extended to more general nonlinearities by McMahon[2] and a simplified and less restrictive proof given by Papoulis.[3] Although Price's theorem is valid for functions of an arbitrary number of joint-Gaussian variables, use of the theorem is essentially limited to functions of only two joint-Gaussian variables. In the general case of n variables applicability of the theorem requires at least the simultaneous solution of a set of $n(n-1)/2$ ordinary differential equations. In this paper, Price's theorem is modified to yield a single ordinary differential equation for the general case. The modified version is motivated by the works of Plackett[4] and Nabeya.[5]

II. A Partial Differential Equation for the Gaussian Distribution

The basis for Price's theorem is actually a partial differential equation satisfied by the multivariate Gaussian distribution. Let $x_1, x_2, \cdots, x_n$ be jointly Gaussian with unit variances, covariance matrix $K_1 = [\rho_{ij}]$ and joint probability density function $p_1(x_1, \cdots, x_n)$. Let $p_\alpha(x_1, \cdots, x_n)$ denote the same probability density function with the covariance matrix K_1 replaced by K_α where

$$K_\alpha = [\alpha^{(1-\delta_{ij})}\rho_{ij}], \tag{1}$$

α is a parameter and δ_{ij} is the Kronecker delta-function. Note that the original density function and covariance matrix are obtained from $p_\alpha(x_1, \cdots, x_n)$ and K_α by setting $\alpha = 1$, and furthermore, that setting $\alpha = 0$ yields $K_0 = I$ and $p_0(x_1, \cdots, x_n) = \prod_{i=1}^{n} p(x_i)$; i.e., the x_i are statistically independent when $\alpha = 0$. Either by a direct calculation[1] or by writing $p_\alpha(x_1, \cdots, x_n)$ as the inverse Fourier transform of its characteristic function, the identity

$$\frac{\partial p_\alpha}{\partial \alpha} = \sum_{i<j} \rho_{ij} \frac{\partial^2 p_\alpha}{\partial x_i \partial x_j} \tag{2}$$

is readily verified. From this equation by repeated differentiation we find

$$\frac{\partial^k p_\alpha}{\partial \alpha^k} = \mathcal{H}^k p_\alpha \tag{3}$$

where the operator $\mathcal{H}$ is defined as

$$\mathcal{H}(\cdot) = \sum_{i<j} \rho_{ij} \frac{\partial^2(\cdot)}{\partial x_i \partial x_j} \tag{4}$$

and $\mathcal{H}^k$ denotes repeated application of the operator k-times. Equation (3) is the fundamental relationship which leads directly to a Price-type theorem.

III. The Modified Version of Price's Theorem

Let $f(x_1, \cdots, x_n)$ be some zero-memory transformation of the variables x_i and define

$$R_\alpha = E_\alpha[f(x_1, \cdots, x_n)], \tag{5}$$

where E_α denotes that the expectation is carried out with respect to the density function $p_\alpha(x_1, \cdots, x_n)$. In general, the problem with which we shall concern ourselves is that of evaluating $R_1 = E_1[f(x_1, \cdots, x_n]$; i.e., R_α for $\alpha = 1$. Multiplying both sides of (3) by $f(x_1, \cdots, x_n)$, integrating over the variables x_i and integrating by parts $2k$ times on the right-hand side yields

$$\frac{d^k R_\alpha}{d\alpha^k} = E_\alpha[\mathcal{H}^k f]. \tag{6}$$

In integrating by parts, we have assumed that products of derivatives of the Gaussian multivariate density and derivatives of the nonlinearity go to zero at $x_i = \pm\infty$, $i = 1, 2, \cdots, n$. This is equivalent to Papoulis' condition on the nonlinearity.[2] Employing the identity[6]

$$\int_a^z dz_1 \int_a^{z_1} dz_2 \cdots \int_a^{z_{k-1}} dz_k\, g(z_k) = \frac{1}{(k-1)!} \int_a^z (z-\xi)^{k-1} g(\xi)\, d\xi \tag{7}$$

[1] See Plackett,[4] eq. (3).
[2] See Papoulis,[3] eq. (3).

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the solution to (6) can be written for $k \geq 1$ as

$$R_1 = \sum_{m=0}^{k-1} \frac{1}{m!} E_0[\mathcal{H}^m f] + \frac{1}{(k-1)!} \int_0^1 (1-\alpha)^{k-1} E_\alpha[\mathcal{H}^k f]\, d\alpha, \tag{8}$$

where we have utilized (6) to evaluate the initial conditions; viz.,

$$\left.\frac{d^m R_\alpha}{d\alpha^m}\right|_{\alpha=0} = E_0[\mathcal{H}^m f]. \tag{9}$$

Either (6) or its solution (8) may be regarded as a modified version of Price's theorem. Note that the initial conditions, the summation in (8), are evaluated by assuming all of the variables $x_1, \cdots, x_n$ to be *statistically independent*.

In the case $n = 2$, (6) reduces to

$$\frac{d^k R_\alpha}{d\alpha^k} = E_\alpha\left[\rho_{12}^k \frac{\partial^{2k}}{\partial x_1^k \partial x_2^k} f(x_1, x_2)\right] \tag{10}$$

which is, upon replacing $\alpha\rho_{12}$ by ρ, McMahon's extension of Price's theorem.[3]

IV. Examples[4]

Example 1—Product Moments

Let $n = 2m$ be an even integer, $f(x_1, \cdots, x_n) = \prod_{i=1}^n x_i$ and take $k = m + 1$. Since $\mathcal{H}^k$ is an operator involving $2k$ differentiations, it follows by inspection that

$$\mathcal{H}^{m+1} f = E_0[f] = E_0[\mathcal{H} f] = \cdots = E_0[\mathcal{H}^{m-1} f] = 0, \tag{11}$$

and (8) yields

$$E_1[x_1 x_2 \cdots x_{2m}] = \frac{1}{m!} E_0[\mathcal{H}^m f]. \tag{12}$$

The only terms of $\mathcal{H}^m f$ which are nonzero are those involving one differentiation with respect to each x_i. Since each of these terms occurs $m!$ times, we obtain

$$E_1[x_1 x_2 \cdots x_{2m}] = \sum_{\text{all pairs}} \rho_{ij}\rho_{kl} \cdots \rho_{qr}, \tag{13}$$

where $i \neq k \neq l \neq$, etc., and the total number of terms in the summation is $(2m)!/2^m m!$.[5] When n is odd, a similar argument shows that $E_1[x_1 x_2 \cdots x_n] = 0$. This result along with (13) can be used as the defining relations for a multivariate Gaussian distribution.[7] Consequently, Gaussianness is both necessary and sufficient for the modified version of Price's theorem to hold.

Example 2—Schläfli integral for $E[\text{sgn}\,(x_1x_2x_3x_4)]$

In this case $n = 4$, we take $k = 1$ and $f = \text{sgn}\,(x_1x_2x_3x_4)$. Again $E_0[f] = 0$ and (8) gives

$$\begin{aligned} E_1[\text{sgn}\,(x_1x_2x_3x_4)] &= \int_0^1 E_\alpha[\mathcal{H}^1 f]\, d\alpha, \\ &= 4 \sum_{i<j} \rho_{ij} \int_0^1 E_\alpha[\delta(x_i)\delta(x_j)\,\text{sgn}\,(x_p x_q)]\, d\alpha, \\ &= \frac{2}{\pi} \sum_{i<j} \rho_{ij} \int_0^1 \frac{E_\alpha[\text{sgn}\,(x_p x_q) \mid x_i = 0, x_j = 0]}{\sqrt{1 - \alpha^2 \rho_{ij}^2}}\, d\alpha, \end{aligned} \tag{14}$$

where $i \neq j \neq p \neq q$ and the expectation is now a conditional expectation with respect to the conditional density function $p_\alpha(x_p, x_q \mid x_i = 0, x_j = 0)$. This conditional expectation has the well-known arcsine form[1] and is most conveniently expressed in terms of *partial correlation coefficients*[8] yielding

$$E_1[\text{sgn}\,(x_1x_2x_3x_4)] = \frac{4}{\pi^2} \sum_{i<j} \rho_{ij} \int_0^1 \frac{\sin^{-1} \rho_{pq\cdot ij}(\alpha)}{\sqrt{1 - \alpha^2 \rho_{ij}^2}}\, d\alpha, \tag{15}$$

where $i \neq j \neq p \neq q$ and $\rho_{pq\cdot ij}(\alpha)$ is a partial correlation coefficient. See Plackett[4] and Nabeya[5] for similar derivations of (15).

Example 3—Product Moments of Error Functions

Consider initially the problem of evaluating[6]

$$R_1 = E_1\left[\prod_{i=1}^4 \text{Erf}\left(\frac{x_i}{l}\right)\right]. \tag{16}$$

Taking $k = 1$ in the modified version of Price's theorem and noting that $E_0[f] = 0$, (8) yields

$$R_1 = \frac{2}{\pi} \sum_{i<j} \frac{\rho_{ij}}{l^2} \cdot \int_0^1 E_\alpha\left[\text{Erf}\left(\frac{x_p}{l}\right) \text{Erf}\left(\frac{x_q}{l}\right) \exp\left(-\frac{x_i^2 + x_j^2}{2l^2}\right)\right] d\alpha. \tag{17}$$

Evaluating the expectation, we find after some algebra

$$R_1 = \frac{4}{\pi^2} \sum_{i<j} \rho_{ij} \int_0^{(1+l^2)^{-1}} \frac{\sin^{-1} \rho_{pq\cdot ij}(\alpha)}{\sqrt{1 - \alpha^2 \rho_{ij}^2}}\, d\alpha. \tag{18}$$

Comparing this result with (15), it follows that

$$E_1\left[\prod_{i=1}^4 \text{Erf}\left(\frac{x_i}{l}\right)\right] = E_{1/(1+l^2)}[\text{sgn}\,(x_1x_2x_3x_4)]. \tag{19}$$

This is a generalization of Baum's result for the case of the product of two error function nonlinearities.[7] The above result shows that averages after smooth limiting are the same as averages after hard limiting except for a scale change in all correlation coefficients.

Equation (19) can be generalized to the case of an arbitrary number of Gaussian variates; viz.,

$$E_1\left[\prod_{i=1}^n \text{Erf}\left(\frac{x_i}{l}\right)\right] = E_{1/(1+l^2)}\left[\prod_{i=1}^n \text{sgn}\, x_i\right]. \tag{20}$$

[3] See Price,[1] eq. (20), and McMahon,[2] eq. (5).

[4] In these examples, all Gaussian variates are assumed to have zero means and unit variances.

[5] See Middleton,[7] p. 343.

[6] Erf $z = \sqrt{2/\pi} \int_0^z \exp(-\xi^2/2)\, d\xi$.

[7] See Price,[1] eqs. (28) and (40).

Although this result could be proved using the modified version of Price's theorem, it is more easily established by an obvious extension of Blachman's derivation[9] for the case $n = 2$.

Example 4—Product Moments Involving Hermite Polynomials

Hermite polynomials arise frequently in the study of nonlinear systems with Gaussian inputs since the Gaussian distribution is the weighting function for orthogonality of these polynomials. The pth order Hermite polynomial is here defined as

$$H_p(x) = (-1)^p e^{\frac{x^2}{2}} \frac{d^p}{dx^p} e^{\frac{-x^2}{2}}; \qquad p = 0, 1, \cdots . \quad (21)$$

If x is a Gaussian variate with zero mean and unit variance, it follows from the above definition that

$$E_1[H_p(x)] = \delta_{p0}. \quad (22)$$

Furthermore, since $dH_p(x)/dx = pH_{p-1}(x)$ for $p > 0$, we have

$$E_1\left[\frac{d^q}{dx^q} H_p(x)\right] = p!\delta_{pq}. \quad (23)$$

Products of Three Polynomials

Consider first the expectation of the product of three Hermite polynomial nonlinearities; i.e.,

$$f(x_1, x_2, x_3) = H_a(x_1)H_b(x_2)H_c(x_3), \quad (24)$$

where $a + b + c \equiv 2m$ is assumed to be an even integer since $E_1[f] = 0$ for $a + b + c$ odd. Taking $k = m + 1$ in the modified version of Price's theorem, (8) yields

$$R_1 = \frac{1}{m!} E_0[\mathcal{H}^m f] \quad (25)$$

since $E_0[f] = E_0[\mathcal{H}f] = \cdots = E_0[\mathcal{H}^{m-1}f] = 0$ by virtue of (23) and $\mathcal{H}^{m+1}f = 0$ since $\mathcal{H}^{m+1}$ involves $2m + 2$ differentiations and the degree of f is $2m$. Writing $D_i = \partial/\partial x_i$ and using the definition (4) to expand the operator $\mathcal{H}$, the above equation becomes

$$R_1 = \frac{1}{m!} E_0[(\rho_{12}D_1D_2 + \rho_{13}D_1D_3 + \rho_{23}D_2D_3)^m f], \quad (26)$$

$$= \frac{1}{m!} E_0\left[\sum_{r=0}^{m}\sum_{s=0}^{r} \binom{m}{r}\binom{r}{s} \rho_{12}^{m-r}\rho_{13}^{r-s}\rho_{23}^{s} D_1^{m-s}D_2^{m-r+s}D_3^r f\right]. \quad (27)$$

Employing (23), we see that the only term of the double summation which is nonzero is that for which $m - s = a$, $m - r + s = b$ and $r = c$. Consequently, making these substitutions, we obtain

$$E_1[H_a(x_1)H_b(x_2)H_c(x_3)] = a!b!c! \frac{\rho_{12}^{m-c}}{(m-c)!}\frac{\rho_{13}^{m-b}}{(m-b)!}\frac{\rho_{23}^{m-a}}{(m-a)!} \quad (28)$$

if $a, b, c \leq m$. If a, b, or c is greater than m the expectation is zero.

In the case when $c = 0$, $H_c(x_3) = 1$. Then the condition that the exponents of the correlation coefficients in (28) be non-negative implies that $a = b = m$ and (28) yields the familiar expression

$$E_1[H_a(x_1)H_b(x_2)] = a!\rho_{12}^a\delta_{ab}. \quad (29)$$

In the special case $a = b = c = n$, (28) gives the simple form

$$E_1[H_n(x_1)H_n(x_2)H_n(x_3)] = \begin{cases} \left[\frac{n!}{(n/2)!}\right]^3 (\rho_{12}\rho_{13}\rho_{23})^{n/2}; & n \text{ even}, \\ 0 \quad ; & n \text{ odd}. \end{cases} \quad (30)$$

The Product of Four Polynomials

Although the modified version of Price's theorem can in principle be applied to evaluate expectations of products of an arbitrary number of Hermite polynomials, the algebra involved soon becomes untractable. As our final example, we consider the case of four identical nonlinearities; viz.,

$$f(x_1, x_2, x_3, x_4) = \prod_{i=1}^{4} H_n(x_i). \quad (31)$$

Taking $k = m + 1$, $m = 2n$, in the modified version of Price's theorem, it follows that (25) is satisfied for the above nonlinearity. Following the above steps leading to (28), we find

$$E_1\left[\prod_{i=1}^{4} H_n(x_i)\right] = (n!)^2 \sum_{r=0}^{n}\sum_{s=0}^{r} \binom{n}{r}^2\binom{r}{s}^2 (\rho_{12}\rho_{34})^{n-r}(\rho_{13}\rho_{24})^{r-s}(\rho_{14}\rho_{23})^s. \quad (32)$$

Comparing this equation with the identity

$$(\alpha + \beta + \gamma)^n = \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}\alpha^{n-r}\beta^{r-s}\gamma^s \quad (33)$$

we can write

$$E_1\left[\prod_{i=1}^{4} H_n(x_i)\right] \doteqdot (n!)^2(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23})^n, \quad (34)$$

where the symbolism $\doteqdot$ means that the binomial coefficients occurring on the right-hand side of (34) are to be squared. Thus

$$E_1\left[\prod_{i=1}^{4} H_n(x_i)\right] \doteqdot (n!)^2\left(E_1\left[\prod_{i=1}^{4} x_i\right]\right)^n. \quad (35)$$

V. Other Modifications

The key to the modified version of Price's theorem is the introduction of the parameter α multiplying all nondiagonal elements in the covariance matrix. However, this parameter could have been introduced in a variety of different ways; for example, two possibilities in the four dimensional case are

$$K_\beta = \begin{bmatrix} 1 & \beta\rho_{12} & \beta\rho_{13} & \beta\rho_{14} \\ \beta\rho_{12} & 1 & \rho_{23} & \rho_{24} \\ \beta\rho_{13} & \rho_{23} & 1 & \rho_{34} \\ \beta\rho_{14} & \rho_{24} & \rho_{34} & 1 \end{bmatrix}$$

and

$$K_\gamma = \begin{bmatrix} 1 & \rho_{12} & \gamma\rho_{13} & \gamma\rho_{14} \\ \rho_{12} & 1 & \gamma\rho_{23} & \gamma\rho_{24} \\ \gamma\rho_{13} & \gamma\rho_{23} & 1 & \rho_{34} \\ \gamma\rho_{14} & \gamma\rho_{24} & \rho_{34} & 1 \end{bmatrix}. \tag{36}$$

For the covariance matrix K_β the partial differential equation analogous to (3) satisfied by the Gaussian distribution p_β is

$$\frac{\partial^k p_\beta}{\partial \beta^k} = \mathcal{H}^k p_\beta, \tag{37}$$

where

$$\mathcal{H}(\cdot) = \sum_{i=2}^{4} \rho_{1i} \frac{\partial^2(\cdot)}{\partial x_1 \partial x_i}. \tag{38}$$

The modified version of Price's theorem in this case is again given by (6), with the solution (8). Note however, that the expectation E_0 in (8) is now carried out assuming only that the variable x_1 is statistically independent of the remaining variables. The above two forms of the covariance matrix have been employed by Nabeya[5] to obtain two other expressions for the Schläfli integral discussed in Example 2.

VI. Summary and Conclusions

This paper has presented a modified version of Price's theorem which reduces to Price's original theorem in the two-dimensional case and is more amenable to analysis in the case of more than two dimensions. The modified version as well as Price's original theorem were shown to follow directly from a partial differential equation satisfied by the Gaussian multivariate density. The first example illustrating the use of the theorem enabled us to evaluate by inspection product moments of Gaussian variates and also demonstrated the necessity of the Gaussian assumption. The next two examples were simple derivations of fourth product moments after hard and smooth limiting of Gaussian variables. The last set of examples considered evaluation of product moments after Hermite polynomial nonlinearities. In particular, the fourth product moment of identical Hermite polynomials was found to be simply related to the fourth product moment of Gaussian variates. Finally, other modifications of Price's theorem were discussed.

In conclusion, we feel that the modified version of Price's theorem provides a simple and useful means for evaluating product moments after nonlinear operations on multidimensional Gaussian variates as does Price's original theorem in the two-dimensional case.

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